High Dimensional Consistent Digital Segments

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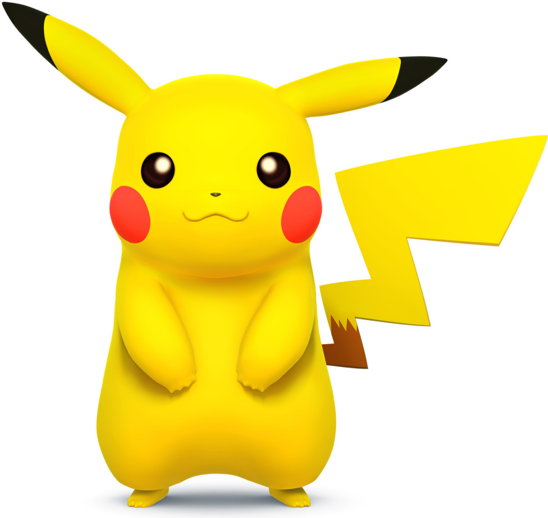
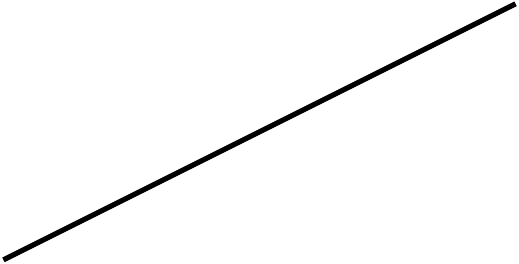
Real life...



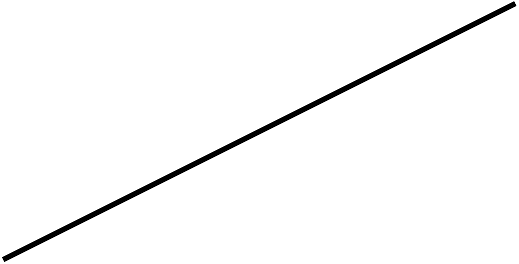
...is digital

Euclidean Segments...



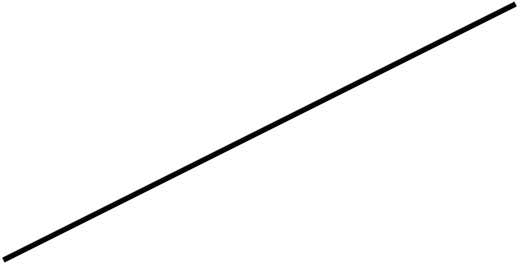
- ...Are beautiful!

Euclidean Segments...



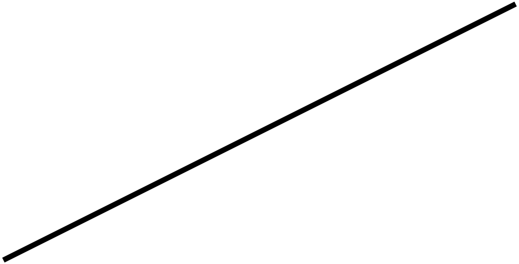
- ...Are beautiful!
- Connected

Euclidean Segments...



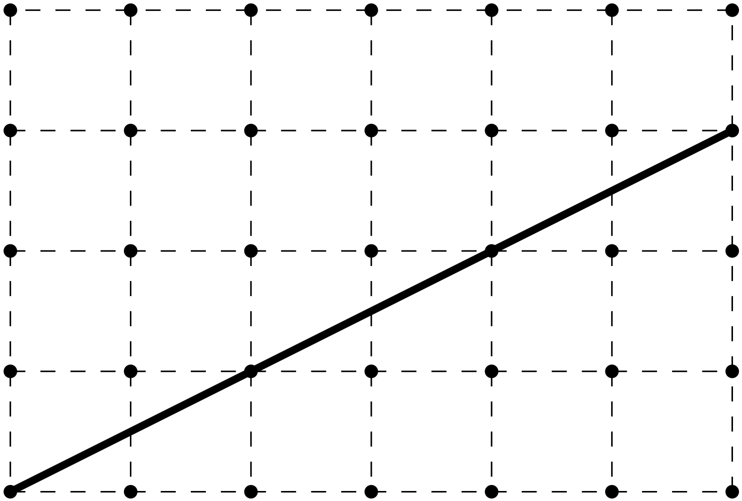
- ...Are beautiful!
- Connected, uniquely defined

Euclidean Segments...



- ...Are beautiful!
- Connected, uniquely defined, can be extended, ...

Digital Segments...



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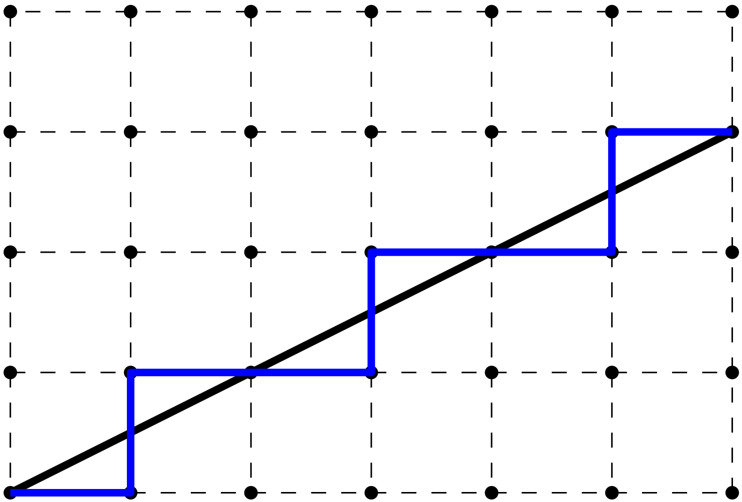

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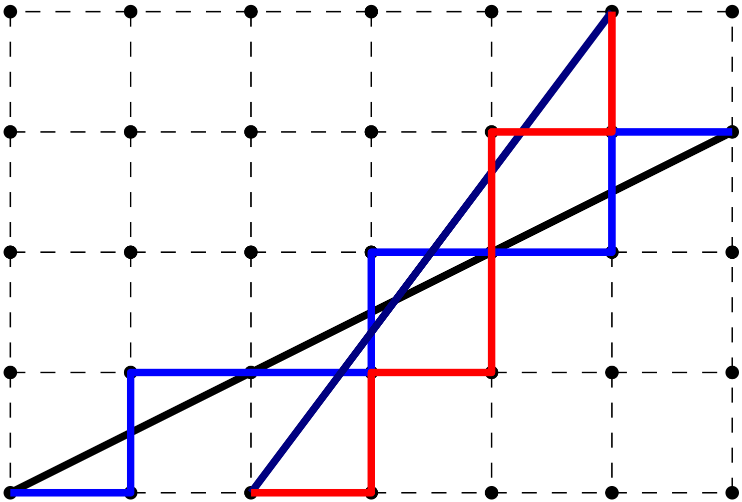

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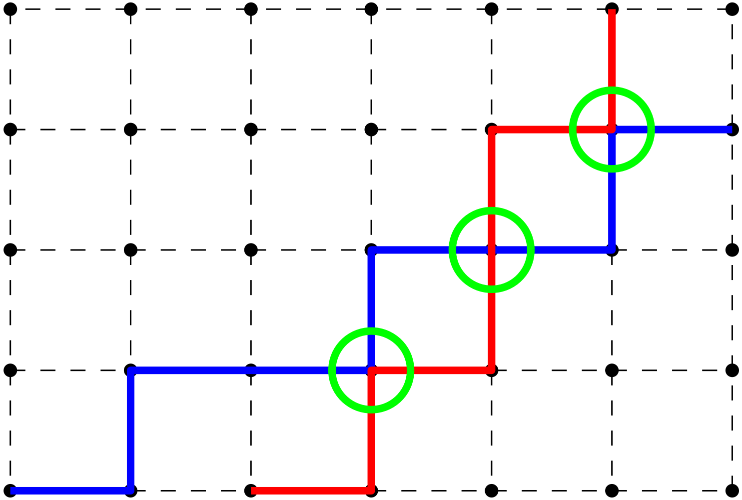

Consistent Digital Segments (CDS)

Digital segments satisfy 5 Axioms :

S1 Grid path property

S2 Symmetry property

S3 Subsegment property

S4 Prolongation property

S5 Monotonicity property

Objective :

- Design digital paths for any two grid points that preserve axioms.

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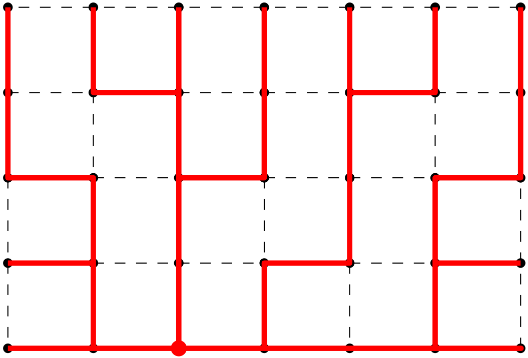
Objective :

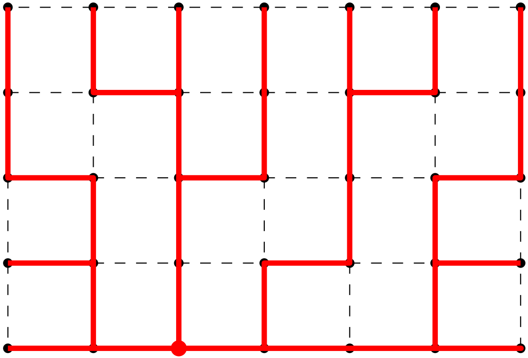
- Design digital paths for any two grid points that preserve axioms.
- Bonus! Can they resemble Euclidean segments?

Intuitive idea



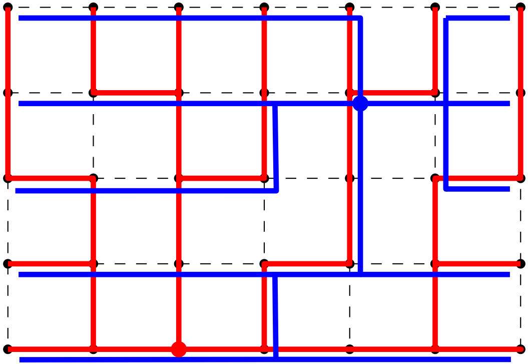
- At a point we need an (infinite) tree.

Intuitive idea



- At a point we need an (infinite) tree. (Consistent Digital Rays (CDR))

Intuitive idea



- At a point we need an (infinite) tree. (Consistent Digital Rays (CDR))
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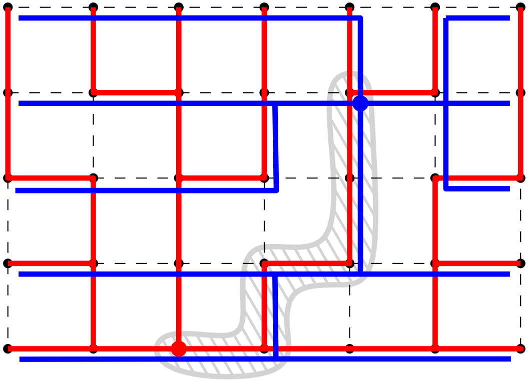

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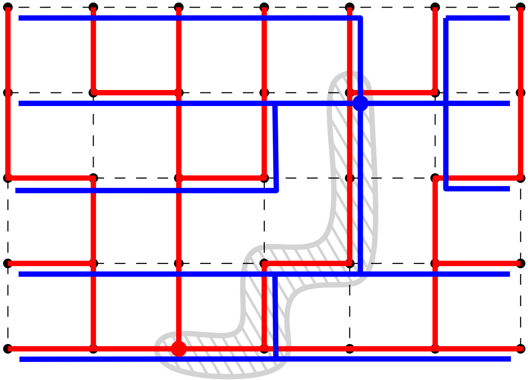

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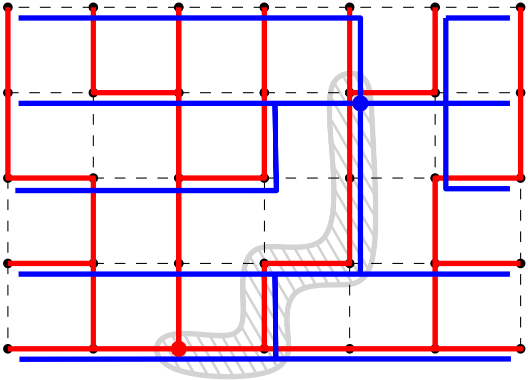
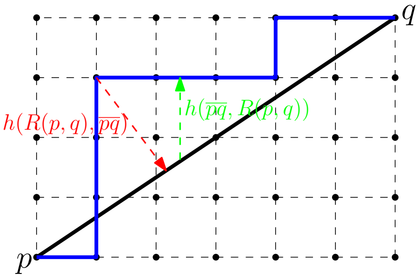


- At a point we need an (infinite) tree. (Consistent Digital Rays (CDR))
- Intersection of two paths must be a path.
- ...also hold with all other trees. (CDS)
- ...and that it resembles Euclidean segments!

How to measure resemblance?



- $h(A, B) = \max_{a \in A} \min_{b \in B} dist(a, b)$.
- The Hausdorff distance between $\overline{pq}$ and $R(p, q)$
  : $H(\overline{pq}, R(p, q)) = \max\{h(\overline{pq}, R(p, q)), h(R(p, q), \overline{pq})\}$.

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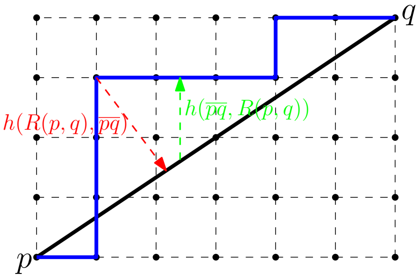

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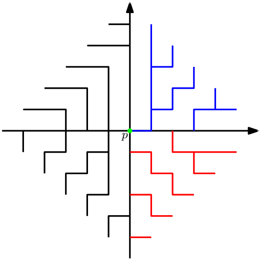
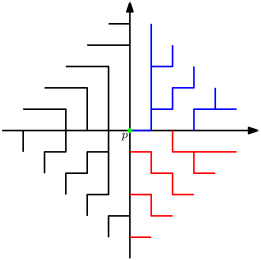
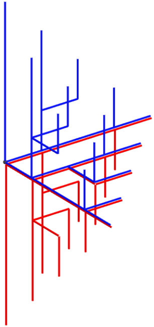
  : $H(\overline{pq}, R(p, q)) = \max\{h(\overline{pq}, R(p, q)), h(R(p, q), \overline{pq})\}$.
- The **Hausdorff distance** of a CDS is defined as $\max_{p,q \in \mathbb{Z}^d, ||p-q||_1 \leq n} H(\overline{pq}, R(p, q))$.

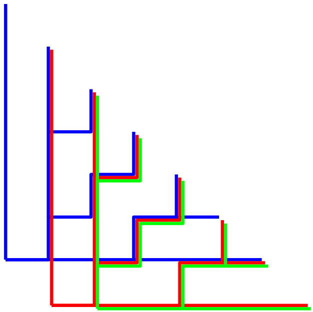
Results

	$(d = 2)$ Previous work	
	a construction from any permutation θ of $\mathbb{Z}$	
CDR (paths from one point)	a θ builds a **quadrant**  **any** 2^d orders make a CDR	
Error	$\Theta(\log n)$ Hausdorff distance	

Results

	$(d = 2)$ Previous work	$(d \geq 3)$ New
	a construction from any permutation θ of $\mathbb{Z}$	generalize the construction to higher dimensions
CDR (paths from one point)	a θ builds a **quadrant**  **any** 2^d orders make a CDR	a θ builds an **orthant**  One order **fixes** the CDR
Error	$\Theta(\log n)$ Hausdorff distance	$\Theta(\log n)$ Hausdorff distance

Our Results

	(d = 2) Previous work	
CDS (paths to/from anywhere)	**any** 2^{d-1} orders make a CDS 	
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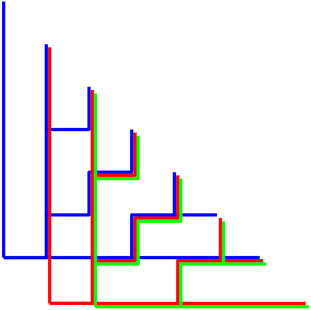
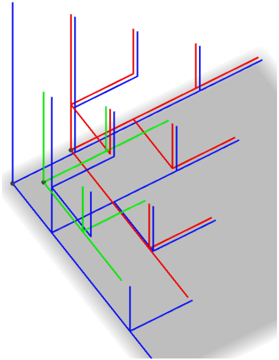
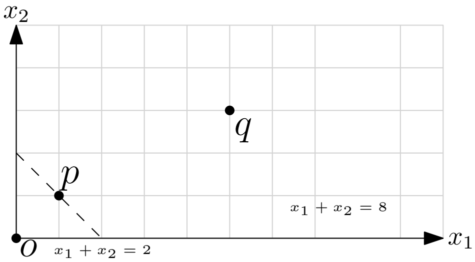
CDS (paths to/from anywhere)	**any** 2^{d-1} orders make a CDS 	**Some** orders work 
Error	$\Theta(\log n)$ Hausdorff distance	$\Theta(n)$ Hausdorff distance

Two-Dimensional Background



- Given a permutation θ, $p = (1, 1)$, $q = (5, 3)$.
- four moves in x_1 direction and 2 moves in x_2 direction.
- Associate each point $m = (m_1, m_2)$ to integer $m_1 + m_2$.

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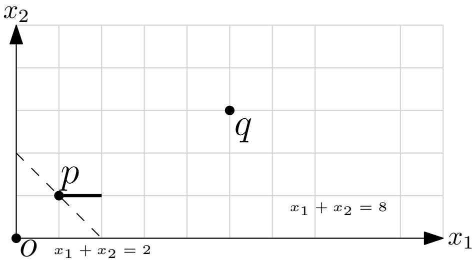

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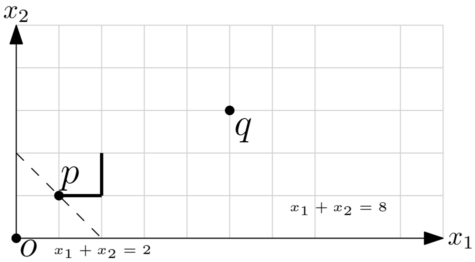
- Look at θ from $p_1 + p_2 = 2$ to $q_1 + q_2 - 1 = 7$.
- $\theta[2, 7] = 7 \prec 6 \prec 4 \prec 2 \prec 3 \prec 5$.
- Look at the position of $m_1 + m_2$ in $\theta[2, 7]$.
- Red numbers represent x_1 movements.



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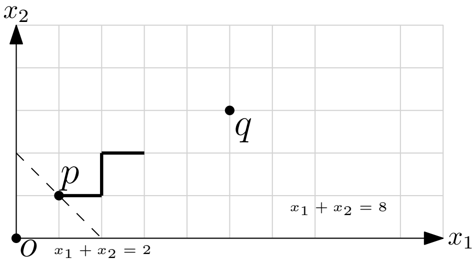

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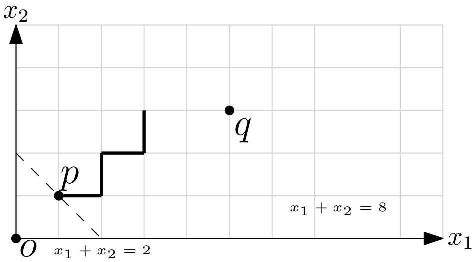

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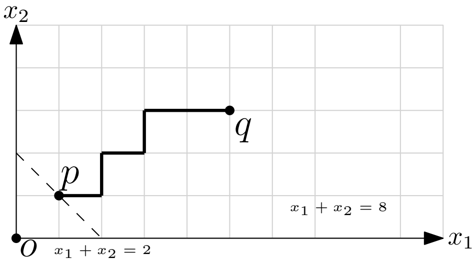

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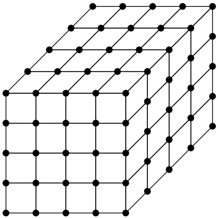
Let's extend this to high dimensions!



- Fix total order θ, construct a path from p to q.

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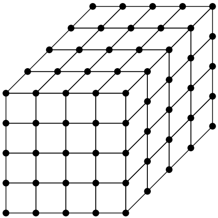

- Fix total order θ, construct a path from p to q.
- We must move $|p_i - q_i|$ times in the x_i-coordinate.

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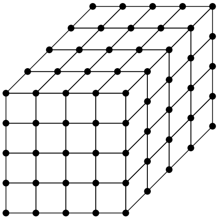

- Fix total order θ, construct a path from p to q.
- We must move $|p_i - q_i|$ times in the x_i-coordinate.
- Partition $\theta[\sum_i p_i, \sum_i q_i - 1]$ into d sub-intervals.

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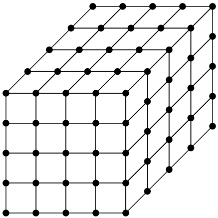

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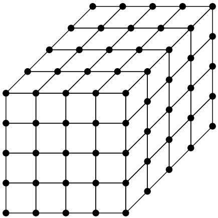

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- We need an axis-order to be consistent.

Axis-order

slope	axis-order
$(+1, +1, +1)$	x_1, x_2, x_3
$(+1, -1, -1)$	x_1, x_3, x_2
$(-1, +1, +1)$	x_2, x_3, x_1
$\dots$	$\dots$

From $(1, 1, 0)$ to $(3, 5, 4)$ slope$= (+1, +1, +1)$

$$\theta[2, 11] = \overbrace{5 \prec 3}^{x_1} \prec \overbrace{2 \prec 7 \prec 9 \prec 8}^{x_2} \prec \overbrace{11 \prec 10 \prec 6 \prec 4}^{x_3}$$

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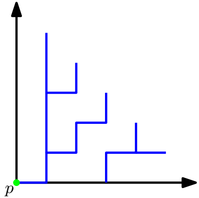
From $(1, 1, 2)$ to $(-3, 5, 4)$ slope$= (-1, +1, +1)$

slope of p, q

$t = (t_1, t_2, \dots, t_d) \in \{+1, -1\}^d$, where $t_i = +1$ if $p_i \leq q_i$ and is -1 if $p_i \geq q_i$.

Necessary and sufficient condition for CDRs



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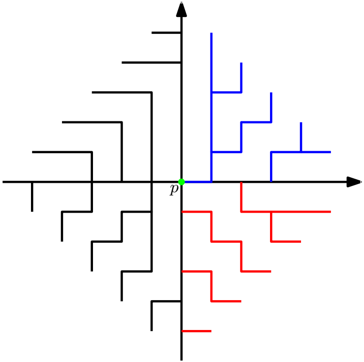
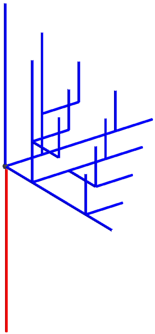


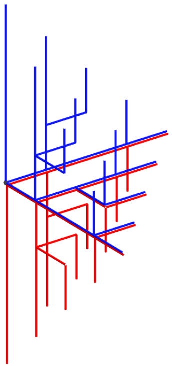
- In $\mathbb{Z}^2$ a total order determines a quadrant
- Any 4 total orders combined form a CDR

Necessary and sufficient condition for CDRs



- Fix $p \in Z^d$, total order θ, and slope t, create an orthant
- Denoted by $TOC(\theta, p, \mathrm{t})$.

Necessary and sufficient condition for CDRs



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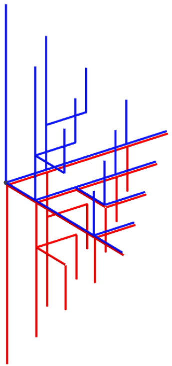

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- **Question** do we always make a CDR?

Necessary and sufficient condition for CDRs

Theorem

$\bigcup_{\mathrm{t}\in T} TOC(\theta_{\mathrm{t}}, p, \mathrm{t})$ is a CDR if and only if $\theta_{\mathrm{t}} = \theta_{\mathrm{t}'} - \mathrm{t}' \cdot p + \mathrm{t} \cdot p$.

Necessary and sufficient condition for CDRs

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Example

- $p = (2, 2, 0)$, $\mathrm{t} = (+1, +1, +1)$, $\mathrm{t}' = (+1, -1, -1)$
- $\mathrm{t} \cdot p = 4$, $\mathrm{t}' \cdot p = 0$, $-\mathrm{t}' \cdot p + \mathrm{t} \cdot p = 4$

Necessary and sufficient condition for CDRs

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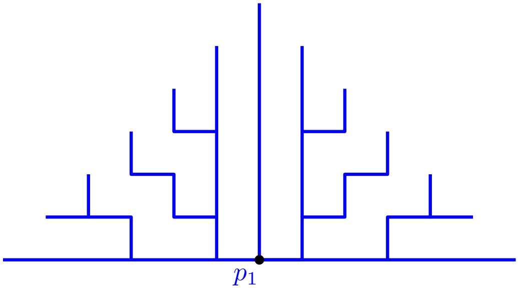
$$\begin{aligned} \theta_{\mathrm{t}'} &= \ldots \prec 5 \prec 4 \prec 6 \prec \ldots \\ \theta_{\mathrm{t}} = \theta_{\mathrm{t}'} + \mathbf{4} &= \ldots \prec 9 \prec 8 \prec 10 \prec \ldots \end{aligned}$$

What about CDSs?



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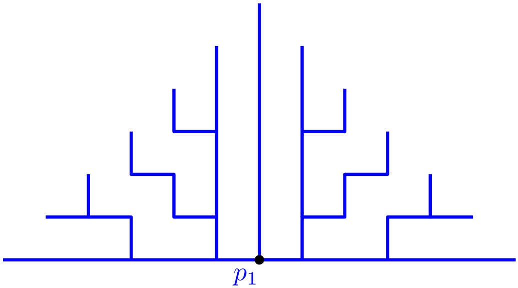

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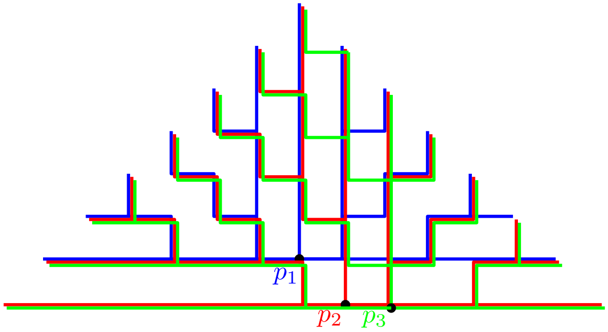

- Look at CDS now (paths for **any** pair of points)
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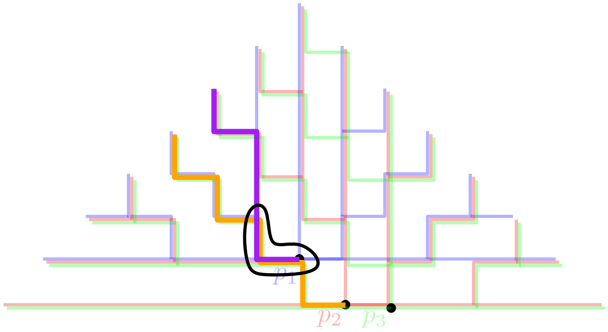
- Repeat at all points to get a CDS

What about CDSs?



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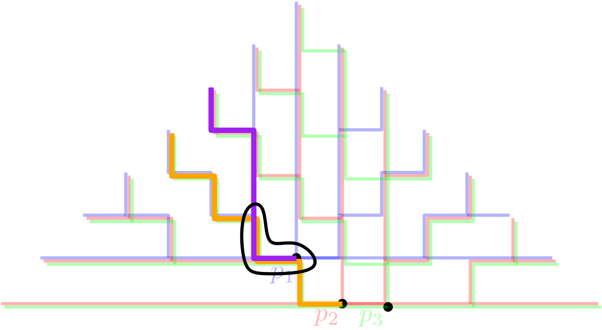

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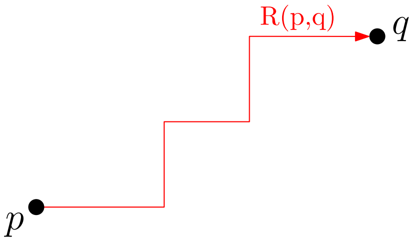

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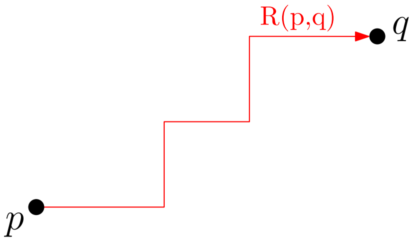

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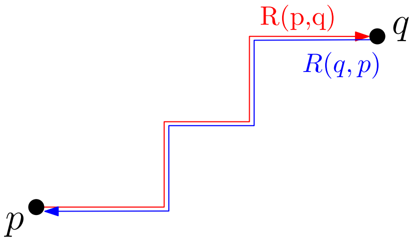

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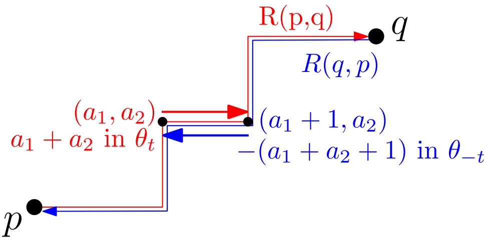

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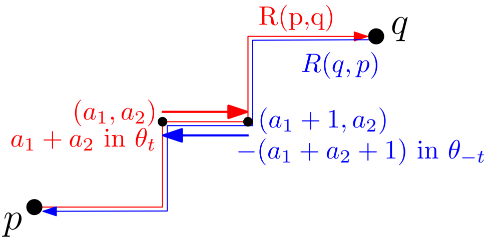

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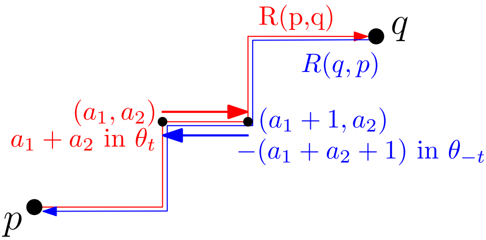

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- The axis-order of $-t$ is the reverse axis-order of t

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Results

- Extended the approach of Christ *et al* to $\mathbb{Z}^d$.
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Open Problems

- Different approach to make CDSs?
- $o(n)$ Hausdorff distance?
- Fully characterize CDRs/CDSs in $\mathbb{Z}^d$?

Questions?Comments?

Thank you!