Near-Feasible Stable Matchings with Budget Constraints (IJCAI'17)

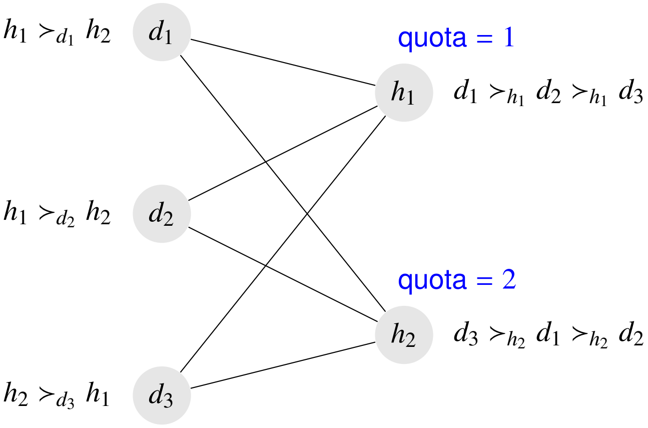
河瀬 康志 (東京工業大学, 理研 AIP)
岩崎 敦 (電気通信大学, 理研 AIP)

August 4, 2017
ERATO 感謝祭 SeasonIV

Outline

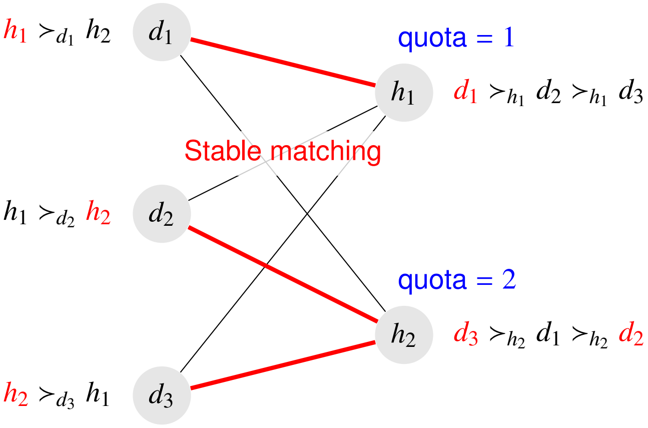
1. Model
2. Generalized Deferred Acceptance Mechanism
3. Mechanisms
   - Strategy-proof mechanism
   - Not strategy-proof mechanism
4. Conclusion

Stable matching with maximum quotas



Stable matchings always exist

Stable matching with maximum quotas



Stable matchings always exist

Our setting

contract (d, h, w): hospital h offers wage w (> 0) to doctor d

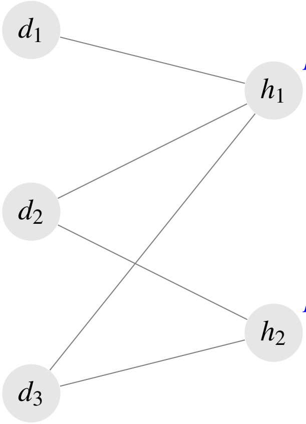
each hospital has an additive utility

possible contracts: $\{(d_1, h_1, 6), (d_1, h_1, 3), (d_2, h_1, 5), (d_2, h_2, 6), (d_3, h_1, 4), (d_3, h_2, 4)\}$

$(d_1, h_1, 6) \succ_{d_1} (d_1, h_1, 3)$

$(d_2, h_1, 5) \succ_{d_2} (d_2, h_2, 6)$

$(d_3, h_2, 4) \succ_{d_3} (d_3, h_1, 4)$



$B_{h_1} = 10$

$(d_1, h_1, 6): \ 8$
$(d_1, h_1, 3): \ 2$
$(d_2, h_1, 5): \ 6$
$(d_3, h_1, 4): \ 4$

$B_{h_2} = 10$

$(d_2, h_2, 6): \ 4$
$(d_3, h_2, 4): \ 5$

Our setting

contract (d, h, w): hospital h offers wage w (> 0) to doctor d

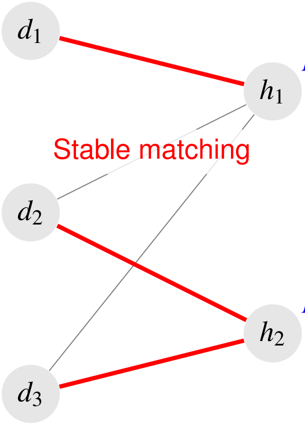
each hospital has an additive utility

possible contracts: $\{(d_1, h_1, 6), (d_1, h_1, 3), (d_2, h_1, 5), (d_2, h_2, 6), (d_3, h_1, 4), (d_3, h_2, 4)\}$

$(d_1, h_1, 6) \succ_{d_1} (d_1, h_1, 3)$

$(d_2, h_1, 5) \succ_{d_2} (d_2, h_2, 6)$

$(d_3, h_2, 4) \succ_{d_3} (d_3, h_1, 4)$



$B_{h_1} = 10$

$(d_1, h_1, 6): 8$

$(d_1, h_1, 3): 2$

$(d_2, h_1, 5): 6$

$(d_3, h_1, 4): 4$

$B_{h_2} = 10$

$(d_2, h_2, 6): 4$

$(d_3, h_2, 4): 5$

Model

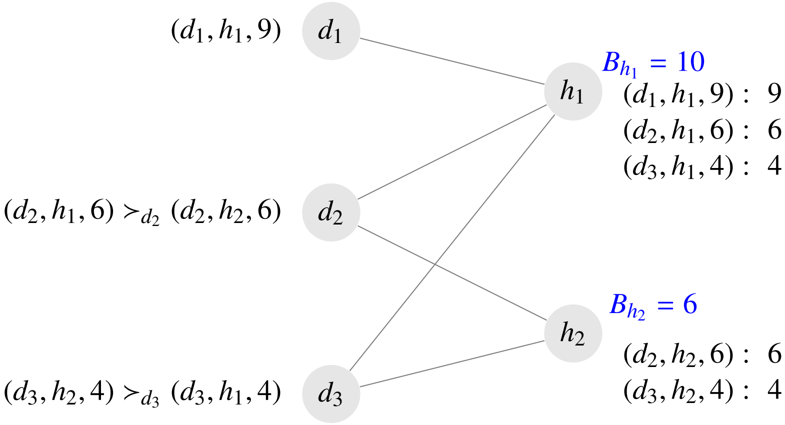
Market $(D, H, X, \succ_D, f_H, B_H)$

- $D = \{d_1, \ldots, d_n\}$: a finite set of doctors
- $H = \{h_1, \ldots, h_m\}$: a finite set of hospitals
- $X \subseteq D \times H \times \mathbb{R}_{++}$: a finite set of contracts
  - $(d, h, w) \in X$: hospital $h \in H$ offers wage $w \in \mathbb{R}_{++}$ to doctor d
  - X'_d: $\{(d', h', w') \in X' \mid d' = d\}$
  - X'_h: $\{(d', h', w') \in X' \mid h' = h\}$
  - x_D, x_H, x_W: the doctor, the hospital, and the wage associated with $x \in X$
- $\succ_D = (\succ_d)_{d \in D}$: a set of strict relations of $d \in D$ over $X_d \cup \{\emptyset\}$
- $f_H = (f_h)_{h \in H}$: a set of hospitals' utility
  - $f_h : X_h \to \mathbb{R}_{++}$: h prefers X' to X'' iff $f_h(X') > f_h(X'')$ $(X', X'' \subseteq X_h)$
  - Assumption: f_h is additive ($f_h(X') = \sum_{x \in X'} f_h(x)$)
- $B_H = (B_h)_{h \in H}$: a set of hospitals' budget

Stable matching

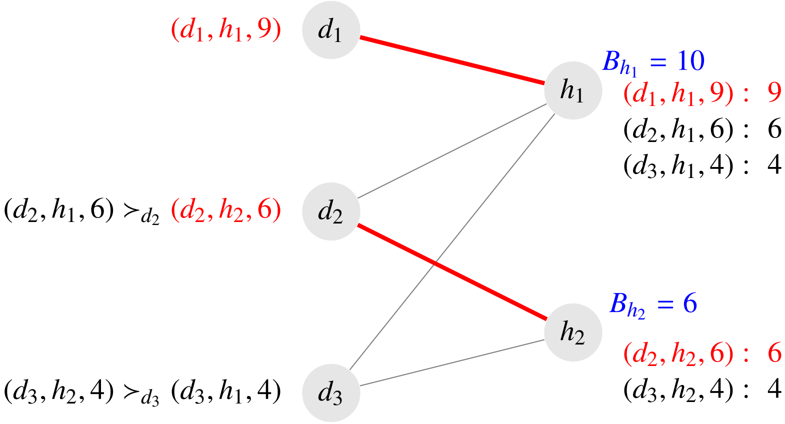
- A subset of contracts $X' \subseteq X$ is a matching if $|X'_d| \leq 1$ $(\forall d \in D)$
- A matching $X' \subseteq X$ is feasible if $w_h(X'_h) \leq B_h$ $(\forall h \in H)$
- $X'' \subseteq X_h$ is a blocking coalition for a matching X' if
  - $X'' \succ_{x_D} X'$ $(\forall x \in X'' \setminus X')$
  - $f_h(X'') > f_h(X'_h)$
- A matching X' is stable if
  - feasible
  - $\nexists$ feasible blocking coalition

Example: no conventional stable matching



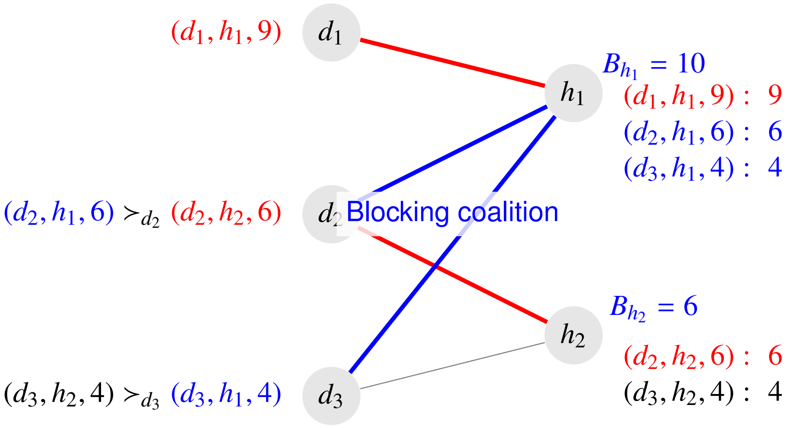
- stable matchings may not exist
- the existence problem is Σ_2^P-complete [Hamada et al. 2017]

Example: no conventional stable matching



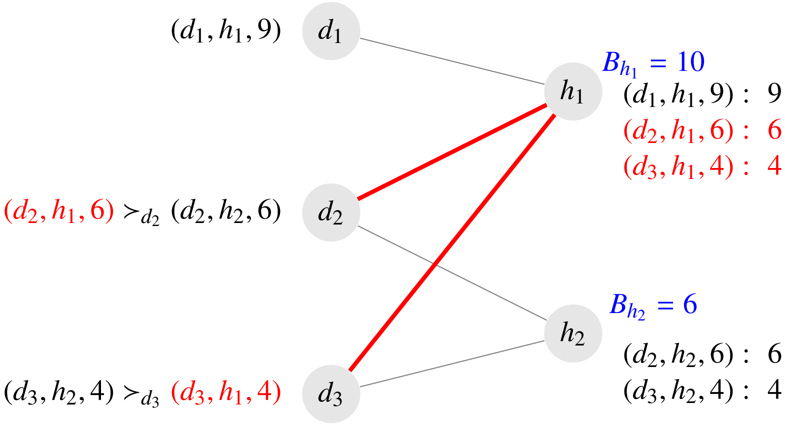
- stable matchings may not exist
- the existence problem is Σ_2^P-complete [Hamada et al. 2017]

Example: no conventional stable matching



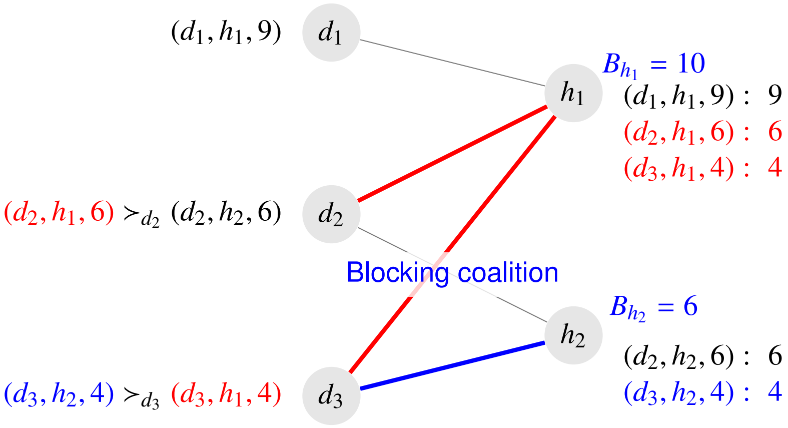
- stable matchings may not exist
- the existence problem is Σ_2^P-complete [Hamada et al. 2017]

Example: no conventional stable matching



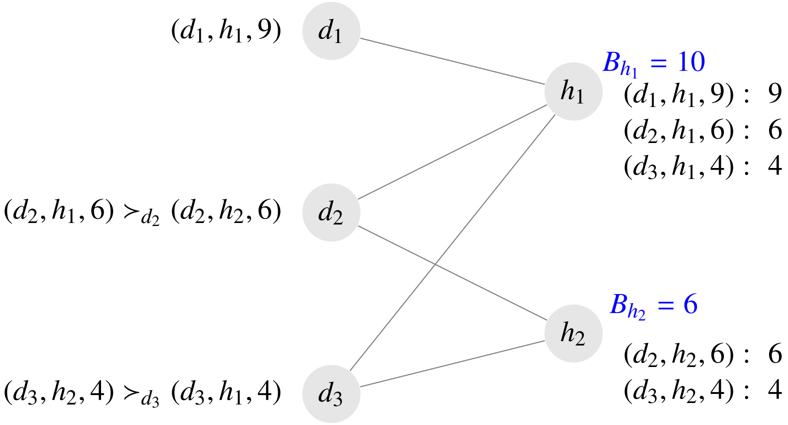
- stable matchings may not exist
- the existence problem is Σ_2^P-complete [Hamada et al. 2017]

Example: no conventional stable matching



- stable matchings may not exist
- the existence problem is Σ_2^P-complete [Hamada et al. 2017]

Example: no conventional stable matching



- stable matchings may not exist
- the existence problem is Σ_2^P-complete [Hamada et al. 2017]

Our approach: B'_H-stable matching

allow a central planner to add or redistribute the budgets ($B_H \to B'_H$)

- A subset of contracts $X' \subseteq X$ is a matching if $|X'_d| \leq 1$ $(\forall d \in D)$
- A matching $X' \subseteq X$ is B'_H-feasible if $w_h(X'_h) \leq B'_h$ $(\forall h \in H)$
- $X'' \subseteq X_h$ is a blocking coalition for a matching X' if
  - $X'' \succ_{x_D} X'$ $(\forall x \in X'' \setminus X')$
  - $f_h(X'') > f_h(X'_h)$
- A matching X' is B'_H-stable if
  - B'_H-feasible
  - $\nexists$ B'_H-feasible blocking coalition

Main results

Theorem

$0 < \forall\alpha < \forall\beta < 1$, $\exists$*market* $(D, H, X, \succ_D, f_H, B_H)$ *such that*

- $x_W \leq \beta \cdot B_{x_H}$ $(\forall x \in X)$ *and*
- $\nexists$ B'_H*-stable matching if* $B_h \leq B'_h \leq (1+\alpha)B_h$ $(\forall h \in H)$

Theorem

$\exists$*mechanism such that*

- *strategy-proof for doctors and*
- *it provides a* B'_H*-stable matching such that*

$$B_h \leq B'_h \leq \left(\max_{x \in X_h} x_W\right) \cdot \left\lceil \frac{B_h}{\min_{x \in X_h} x_W} \right\rceil \quad (\forall h \in H)$$

Theorem

$\exists$*mechanism that provides a* B'_H*-stable matching such that*

$$B_h \leq B'_h < B_h + \max_{x \in X_h} x_W \quad (\forall h \in H)$$

Outline

GDA mechanism [Hatfield–Milgrom 2005]

- $\mathrm{Ch}_D : 2^X \to 2^X$
  - $\mathrm{Ch}_d(X') = \begin{cases} \{x\} & (\text{if } x \in X'_d \text{ and } x \succeq_d x' \ \forall x' \in X'_d \cup \{\emptyset\}) \\ \emptyset & (\text{if } \emptyset \succ_d x' \text{ for all } x' \in X'_d) \end{cases}$
  - $\mathrm{Ch}_D(X') := \bigcup_{d \in D} \mathrm{Ch}_d(X'_d)$
- $\mathrm{Ch}_H : 2^X \to 2^X$
  - $\mathrm{Ch}_H(X') := \bigcup_{h \in H} \mathrm{Ch}_h(X'_h)$ where Ch_h is a choice function of h
  - We have some flexibility when we are allowed to violate budgets

Generalized Deferred Acceptance (GDA)

input: $X, \mathrm{Ch}_D, \mathrm{Ch}_H$ **output:** matching $X' \subseteq X$

$R \leftarrow \emptyset$;

for $i = 1, 2, \ldots$ **do**

  $Y \leftarrow \mathrm{Ch}_D(X \setminus R)$;

  $Z \leftarrow \mathrm{Ch}_H(Y)$;

  $R \leftarrow R \cup (Y \setminus Z)$;

  if $Y = Z$ **then return** Y;

Important properties

Substitutability (SUB)

$$X'' \subseteq X' \subseteq X \Longrightarrow \mathrm{Ch}_H(X') \cap X'' \subseteq \mathrm{Ch}_H(X'')$$

Irrelevance of Rejected Contracts (IRC)

$$\mathrm{Ch}_H(X'') \subseteq X' \subseteq X'' \subseteq X \Longrightarrow \mathrm{Ch}_H(X') = \mathrm{Ch}_H(X'')$$

Law of Aggregate Demand (LAD)

$$X'' \subseteq X' \subseteq X \Longrightarrow |\mathrm{Ch}_H(X'')| \leq |\mathrm{Ch}_H(X')|$$

Theorem (Hatfield and Milgrom 2005)

- Ch_H *satisfies SUB and IRC* $\Rightarrow$ *GDA produces* X' *such that*
  - $X' = \mathrm{Ch}_D(X') = \mathrm{Ch}_H(X')$
  - $\nexists X'' \subseteq X_h$ *such that* $X'' \neq \mathrm{Ch}_h(X'_h)$ *and* $X'' = \mathrm{Ch}_h(X'_h \cup X'') \subseteq \mathrm{Ch}_D(X' \cup X'')$
- Ch_H *further satisfies LAD* $\Rightarrow$ *GDA is strategy-proof for doctors*

New property: Compatibility

- Choice functions should maximize the hospital's utility
- We allow choice functions to violate budget constraints

Definition (Compatibility (COM))

$$\mathrm{Ch}_h(X') \in \operatorname*{arg\,max}_{X'' \subseteq X' :\ w_h(X'') \leq \max\{B_h, w_h(\mathrm{Ch}_h(X'))\}} f_h(X'') \quad (\forall X' \subseteq X_h)$$

Theorem

Ch_H *satisfies SUB, IRC, and COM* $\Rightarrow$ *GDA produces* X' *such that*

- X' *is* B'_H*-stable*
- $B'_H = (\max\{B_h, w_h(X')\})_{h \in H}$

Direct choice function

$$\mathrm{Ch}_h(X') \in \operatorname*{arg\,max}_{X'' \subseteq X'_h,\ w_h(X'') \le B_h} f_h(X'')$$

- ☺ COM: $f_h(\mathrm{Ch}_h(X')) \in \arg\max_{X'' \subseteq X':\ w_h(X'') \le \max\{B_h, w_h(\mathrm{Ch}_h(X'))\}} f_h(X'')$
- ☹ SUB: $X'' \subseteq X' \subseteq X \Longrightarrow \mathrm{Ch}_H(X') \cap X'' \subseteq \mathrm{Ch}_H(X'')$
  - let $x_i = (d_i, h, i)$, $f_h(x_i) = i$, and $B_h = 5$
  - then $\mathrm{Ch}_h(\{x_3, x_4\}) = \{x_4\}$ but $\mathrm{Ch}_h(\{x_2, x_3, x_4\}) = \{x_2, x_3\}$
- ☺ IRC: $\mathrm{Ch}_H(X'') \subseteq X' \subseteq X'' \subseteq X \Longrightarrow \mathrm{Ch}_H(X') = \mathrm{Ch}_H(X'')$
- ☹ LAD: $X'' \subseteq X' \subseteq X \Longrightarrow |\mathrm{Ch}_H(X'')| \le |\mathrm{Ch}_H(X')|$
- ☺ $w_h(\mathrm{Ch}_h(X')) \le B_h$
- ☹ hard to compute (NP-hard)

Outline

Our mechanisms

We provide the following two mechanisms

1. strategy-proof mechanism that gives a B'_H-stable matching such that

$$B_h \leq B'_h \leq \left(\max_{x \in X_h} x_W \right) \cdot \left\lceil \frac{B_h}{\min_{x \in X_h} x_W} \right\rceil \quad (\forall h \in H)$$

2. not strategy-proof mechanism that gives a B'_H-stable matching such that

$$B_h \leq B'_h < B_h + \max_{x \in X_h} x_W \quad (\forall h \in H)$$

Outline

Strategy-proof mechanism

Consider the GDA with the following choice functions:

$\mathrm{Ch}_h(X')$ for $X' \subseteq X_h$

- Sort X' in descending order of utility per unit wage
- Return top $\min\{\lceil B_h / \min_{x \in X_h} x_W \rceil, |X'|\}$ contracts

- ☺ COM: $f_h(\mathrm{Ch}_h(X')) \in \arg\max_{X'' \subseteq X' :\ w_h(X'') \le \max\{B_h, w_h(\mathrm{Ch}_h(X'))\}} f_h(X'')$
- ☺ SUB: $X'' \subseteq X' \subseteq X \implies \mathrm{Ch}_H(X') \cap X'' \subseteq \mathrm{Ch}_H(X'')$
- ☺ IRC: $\mathrm{Ch}_H(X'') \subseteq X' \subseteq X'' \subseteq X \implies \mathrm{Ch}_H(X') = \mathrm{Ch}_H(X'')$
- ☺ LAD: $X'' \subseteq X' \subseteq X \implies |\mathrm{Ch}_H(X'')| \le |\mathrm{Ch}_H(X')|$
- 😐 $\mathrm{Ch}_h(X') \le (\max_{x \in X_h} x_W) \cdot \lceil B_h / \min_{x \in X_h} x_W \rceil$
- ☺ efficiently computable

$\rightsquigarrow$ the mechanism is strategy-proof and provides a B'_H-stable matching such that

$$B_h \le B'_h \le \left(\max_{x \in X_h} x_W\right) \cdot \left\lceil \frac{B_h}{\min_{x \in X_h} x_W} \right\rceil \quad (\forall h \in H)$$

Strategy-proof mechanism

Consider the GDA with the following choice functions:

$\mathrm{Ch}_h(X')$ for $X' \subseteq X_h$

- Sort X' in descending order of utility per unit wage
- Return top $\min\{\lceil B_h / \min_{x \in X_h} x_W \rceil, |X'|\}$ contracts

- ☺ COM: $f_h(\mathrm{Ch}_h(X')) \in \arg\max_{X'' \subseteq X' :\ w_h(X'') \leq \max\{B_h, w_h(\mathrm{Ch}_h(X'))\}} f_h(X'')$
- ☺ SUB: $X'' \subseteq X' \subseteq X \implies \mathrm{Ch}_H(X') \cap X'' \subseteq \mathrm{Ch}_H(X'')$
- ☺ IRC: $\mathrm{Ch}_H(X'') \subseteq X' \subseteq X'' \subseteq X \implies \mathrm{Ch}_H(X') = \mathrm{Ch}_H(X'')$
- ☺ LAD: $X'' \subseteq X' \subseteq X \implies |\mathrm{Ch}_H(X'')| \leq |\mathrm{Ch}_H(X')|$
- 😐 $\mathrm{Ch}_h(X') \leq (\max_{x \in X_h} x_W) \cdot \lceil B_h / \min_{x \in X_h} x_W \rceil$
- ☺ efficiently computable

$\rightsquigarrow$ the mechanism is strategy-proof and provides a B'_H-stable matching such that

$$B_h \leq B'_h \leq \left(\max_{x \in X_h} x_W \right) \cdot \left\lceil \frac{B_h}{\min_{x \in X_h} x_W} \right\rceil \quad (\forall h \in H)$$

Strategy-proof mechanism

Consider the GDA with the following choice functions:

$\mathrm{Ch}_h(X')$ for $X' \subseteq X_h$

- Sort X' in descending order of utility per unit wage
- Return top $\min\{\lceil B_h / \min_{x \in X_h} x_W \rceil, |X'|\}$ contracts

- ☺ COM: $f_h(\mathrm{Ch}_h(X')) \in \arg\max_{X'' \subseteq X' :\ w_h(X'') \leq \max\{B_h, w_h(\mathrm{Ch}_h(X'))\}} f_h(X'')$
- ☺ SUB: $X'' \subseteq X' \subseteq X \implies \mathrm{Ch}_H(X') \cap X'' \subseteq \mathrm{Ch}_H(X'')$
- ☺ IRC: $\mathrm{Ch}_H(X'') \subseteq X' \subseteq X'' \subseteq X \implies \mathrm{Ch}_H(X') = \mathrm{Ch}_H(X'')$
- ☺ LAD: $X'' \subseteq X' \subseteq X \implies |\mathrm{Ch}_H(X'')| \leq |\mathrm{Ch}_H(X')|$
- 😐 $\mathrm{Ch}_h(X') \leq (\max_{x \in X_h} x_W) \cdot \lceil B_h / \min_{x \in X_h} x_W \rceil$
- ☺ efficiently computable

$\rightsquigarrow$ the mechanism is strategy-proof and provides a B'_H-stable matching such that

$$B_h \leq B'_h \leq \left(\max_{x \in X_h} x_W \right) \cdot \left\lceil \frac{B_h}{\min_{x \in X_h} x_W} \right\rceil \quad (\forall h \in H)$$

Example

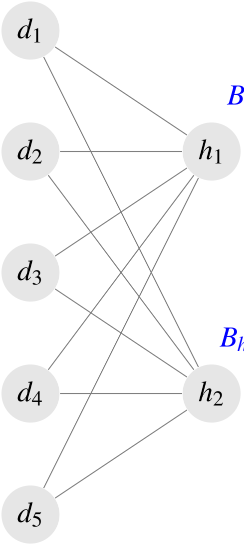
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$, $\lceil 100/42 \rceil = 3$

$(d_1, h_1, 57):\ 111 \approx 57 \cdot 1.95$
$(d_2, h_1, 50):\ 98 = 50 \cdot 1.96$
$(d_3, h_1, 42):\ 83 \approx 42 \cdot 1.98$
$(d_4, h_1, 55):\ 110 = 55 \cdot 2$
$(d_5, h_1, 50):\ 101 = 50 \cdot 2.02$

$B_{h_2} = 100$, $\lceil 100/100 \rceil = 1$

$(d_1, h_2, 100):\ 50$
$(d_2, h_2, 100):\ 30$
$(d_3, h_2, 100):\ 20$
$(d_4, h_2, 100):\ 10$
$(d_5, h_2, 100):\ 40$

Example

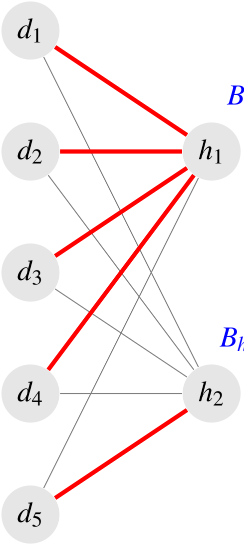
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$, $\lceil 100/42 \rceil = 3$

$(d_1, h_1, 57): \; 111 \approx 57 \cdot 1.95$
$(d_2, h_1, 50): \; 98 = 50 \cdot 1.96$
$(d_3, h_1, 42): \; 83 \approx 42 \cdot 1.98$
$(d_4, h_1, 55): \; 110 = 55 \cdot 2$
$(d_5, h_1, 50): \; 101 = 50 \cdot 2.02$

$B_{h_2} = 100$, $\lceil 100/100 \rceil = 1$

$(d_1, h_2, 100): \; 50$
$(d_2, h_2, 100): \; 30$
$(d_3, h_2, 100): \; 20$
$(d_4, h_2, 100): \; 10$
$(d_5, h_2, 100): \; 40$

Example

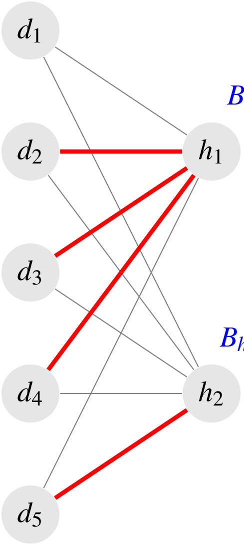
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$, $\lceil 100/42 \rceil = 3$

$(d_1, h_1, 57): \; 111 \approx 57 \cdot 1.95$
$(d_2, h_1, 50): \; 98 = 50 \cdot 1.96$
$(d_3, h_1, 42): \; 83 \approx 42 \cdot 1.98$
$(d_4, h_1, 55): \; 110 = 55 \cdot 2$
$(d_5, h_1, 50): \; 101 = 50 \cdot 2.02$

$B_{h_2} = 100$, $\lceil 100/100 \rceil = 1$

$(d_1, h_2, 100): \; 50$
$(d_2, h_2, 100): \; 30$
$(d_3, h_2, 100): \; 20$
$(d_4, h_2, 100): \; 10$
$(d_5, h_2, 100): \; 40$

Example

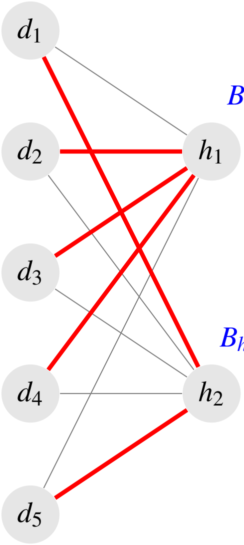
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$, $\lceil 100/42 \rceil = 3$

$(d_1, h_1, 57): \ 111 \approx 57 \cdot 1.95$

$(d_2, h_1, 50): \ 98 = 50 \cdot 1.96$

$(d_3, h_1, 42): \ 83 \approx 42 \cdot 1.98$

$(d_4, h_1, 55): \ 110 = 55 \cdot 2$

$(d_5, h_1, 50): \ 101 = 50 \cdot 2.02$

$B_{h_2} = 100$, $\lceil 100/100 \rceil = 1$

$(d_1, h_2, 100): \ 50$

$(d_2, h_2, 100): \ 30$

$(d_3, h_2, 100): \ 20$

$(d_4, h_2, 100): \ 10$

$(d_5, h_2, 100): \ 40$

Example

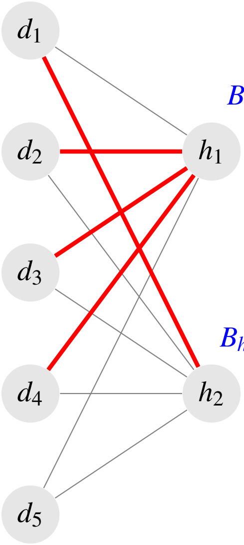
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$, $\lceil 100/42 \rceil = 3$

$(d_1, h_1, 57): \ 111 \approx 57 \cdot 1.95$
$(d_2, h_1, 50): \ 98 = 50 \cdot 1.96$
$(d_3, h_1, 42): \ 83 \approx 42 \cdot 1.98$
$(d_4, h_1, 55): \ 110 = 55 \cdot 2$
$(d_5, h_1, 50): \ 101 = 50 \cdot 2.02$

$B_{h_2} = 100$, $\lceil 100/100 \rceil = 1$

$(d_1, h_2, 100): \ 50$
$(d_2, h_2, 100): \ 30$
$(d_3, h_2, 100): \ 20$
$(d_4, h_2, 100): \ 10$
$(d_5, h_2, 100): \ 40$

Example

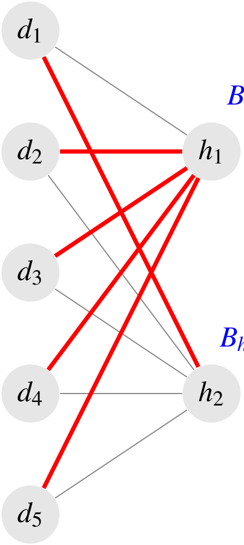
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$, $\lceil 100/42 \rceil = 3$

$(d_1, h_1, 57):\ 111 \approx 57 \cdot 1.95$

$(d_2, h_1, 50):\ 98 = 50 \cdot 1.96$

$(d_3, h_1, 42):\ 83 \approx 42 \cdot 1.98$

$(d_4, h_1, 55):\ 110 = 55 \cdot 2$

$(d_5, h_1, 50):\ 101 = 50 \cdot 2.02$

$B_{h_2} = 100$, $\lceil 100/100 \rceil = 1$

$(d_1, h_2, 100):\ 50$

$(d_2, h_2, 100):\ 30$

$(d_3, h_2, 100):\ 20$

$(d_4, h_2, 100):\ 10$

$(d_5, h_2, 100):\ 40$

Example

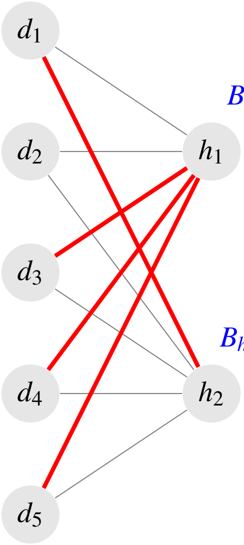
$(d_1, h_1, 57) \succ_{d_1}$ $(d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5}$ $(d_5, h_1, 50)$



$B_{h_1} = 100$, $\lceil 100/42 \rceil = 3$

$(d_1, h_1, 57): \ 111 \approx 57 \cdot 1.95$
$(d_2, h_1, 50): \ 98 = 50 \cdot 1.96$
$(d_3, h_1, 42): \ 83 \approx 42 \cdot 1.98$
$(d_4, h_1, 55): \ 110 = 55 \cdot 2$
$(d_5, h_1, 50): \ 101 = 50 \cdot 2.02$

$B_{h_2} = 100$, $\lceil 100/100 \rceil = 1$

$(d_1, h_2, 100): \ 50$
$(d_2, h_2, 100): \ 30$
$(d_3, h_2, 100): \ 20$
$(d_4, h_2, 100): \ 10$
$(d_5, h_2, 100): \ 40$

Example

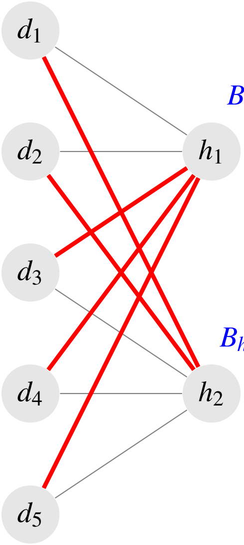
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$, $\lceil 100/42 \rceil = 3$

$(d_1, h_1, 57): \; 111 \approx 57 \cdot 1.95$

$(d_2, h_1, 50): \; 98 = 50 \cdot 1.96$

$(d_3, h_1, 42): \; 83 \approx 42 \cdot 1.98$

$(d_4, h_1, 55): \; 110 = 55 \cdot 2$

$(d_5, h_1, 50): \; 101 = 50 \cdot 2.02$

$B_{h_2} = 100$, $\lceil 100/100 \rceil = 1$

$(d_1, h_2, 100): \; 50$

$(d_2, h_2, 100): \; 30$

$(d_3, h_2, 100): \; 20$

$(d_4, h_2, 100): \; 10$

$(d_5, h_2, 100): \; 40$

Example

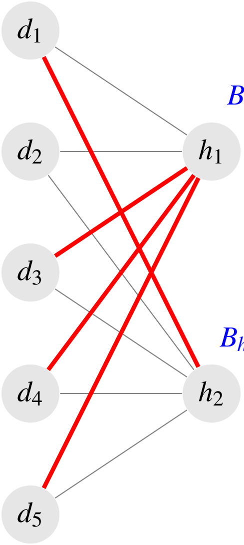
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$, $\lceil 100/42 \rceil = 3$

$(d_1, h_1, 57): \; 111 \approx 57 \cdot 1.95$

$(d_2, h_1, 50): \; 98 = 50 \cdot 1.96$

$(d_3, h_1, 42): \; 83 \approx 42 \cdot 1.98$

$(d_4, h_1, 55): \; 110 = 55 \cdot 2$

$(d_5, h_1, 50): \; 101 = 50 \cdot 2.02$

$B_{h_2} = 100$, $\lceil 100/100 \rceil = 1$

$(d_1, h_2, 100): \; 50$

$(d_2, h_2, 100): \; 30$

$(d_3, h_2, 100): \; 20$

$(d_4, h_2, 100): \; 10$

$(d_5, h_2, 100): \; 40$

Outline

Not strategy-proof mechanism

Consider the GDA with the following choice functions:

$\mathrm{Ch}_h(X')$ for $X' \subseteq X_h$

- Sort X' in descending order of utility per unit wage
- Greedily return contracts until the total wage exceeds the budget

- ☺ COM: $f_h(\mathrm{Ch}_h(X')) \in \arg\max_{X'' \subseteq X' \colon w_h(X'') \leq \max\{B_h, w_h(\mathrm{Ch}_h(X'))\}} f_h(X'')$
- ☺ SUB: $X'' \subseteq X' \subseteq X \implies \mathrm{Ch}_H(X') \cap X'' \subseteq \mathrm{Ch}_H(X'')$
- ☺ IRC: $\mathrm{Ch}_H(X'') \subseteq X' \subseteq X'' \subseteq X \implies \mathrm{Ch}_H(X') = \mathrm{Ch}_H(X'')$
- ☹ LAD: $X'' \subseteq X' \subseteq X \implies |\mathrm{Ch}_H(X'')| \leq |\mathrm{Ch}_H(X')|$
- ☺ $\mathrm{Ch}_h(X') < B_h + \max_{x \in X_h} x_W$
- ☺ efficiently computable

$\rightsquigarrow$ the mechanism provides a B'_H-stable matching such that

$$B_h \leq B'_h < B_h + \max_{x \in X_h} x_W \quad (\forall h \in H)$$

Not strategy-proof mechanism

Consider the GDA with the following choice functions:

$\mathrm{Ch}_h(X')$ for $X' \subseteq X_h$

- Sort X' in descending order of utility per unit wage
- Greedily return contracts until the total wage exceeds the budget

- ☺ COM: $f_h(\mathrm{Ch}_h(X')) \in \arg\max_{X'' \subseteq X' \colon w_h(X'') \leq \max\{B_h, w_h(\mathrm{Ch}_h(X'))\}} f_h(X'')$
- ☺ SUB: $X'' \subseteq X' \subseteq X \implies \mathrm{Ch}_H(X') \cap X'' \subseteq \mathrm{Ch}_H(X'')$
- ☺ IRC: $\mathrm{Ch}_H(X'') \subseteq X' \subseteq X'' \subseteq X \implies \mathrm{Ch}_H(X') = \mathrm{Ch}_H(X'')$
- ☹ LAD: $X'' \subseteq X' \subseteq X \implies |\mathrm{Ch}_H(X'')| \leq |\mathrm{Ch}_H(X')|$
- ☺ $\mathrm{Ch}_h(X') < B_h + \max_{x \in X_h} x_W$
- ☺ efficiently computable

$\rightsquigarrow$ the mechanism provides a B'_H-stable matching such that

$$B_h \leq B'_h < B_h + \max_{x \in X_h} x_W \quad (\forall h \in H)$$

Not strategy-proof mechanism

Consider the GDA with the following choice functions:

$\mathrm{Ch}_h(X')$ for $X' \subseteq X_h$

- Sort X' in descending order of utility per unit wage
- Greedily return contracts until the total wage exceeds the budget

- ☺ COM: $f_h(\mathrm{Ch}_h(X')) \in \arg\max_{X'' \subseteq X' \colon w_h(X'') \leq \max\{B_h, w_h(\mathrm{Ch}_h(X'))\}} f_h(X'')$
- ☺ SUB: $X'' \subseteq X' \subseteq X \Longrightarrow \mathrm{Ch}_H(X') \cap X'' \subseteq \mathrm{Ch}_H(X'')$
- ☺ IRC: $\mathrm{Ch}_H(X'') \subseteq X' \subseteq X'' \subseteq X \Longrightarrow \mathrm{Ch}_H(X') = \mathrm{Ch}_H(X'')$
- ☹ LAD: $X'' \subseteq X' \subseteq X \Longrightarrow |\mathrm{Ch}_H(X'')| \leq |\mathrm{Ch}_H(X')|$
- ☺ $\mathrm{Ch}_h(X') < B_h + \max_{x \in X_h} x_W$
- ☺ efficiently computable

$\rightsquigarrow$ the mechanism provides a B'_H-stable matching such that

$$B_h \leq B'_h < B_h + \max_{x \in X_h} x_W \quad (\forall h \in H)$$

Example

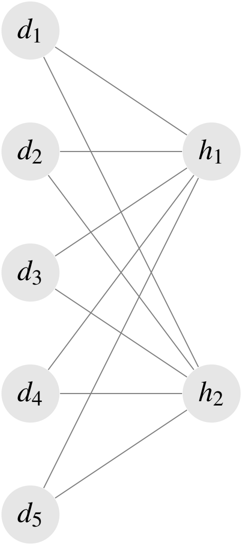
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$

$(d_1, h_1, 57):\ 111 \approx 57 \cdot 1.95$
$(d_2, h_1, 50):\ 98 = 50 \cdot 1.96$
$(d_3, h_1, 42):\ 83 \approx 42 \cdot 1.98$
$(d_4, h_1, 55):\ 110 = 55 \cdot 2$
$(d_5, h_1, 50):\ 101 = 50 \cdot 2.02$

$B_{h_2} = 100$

$(d_1, h_2, 100):\ 50$
$(d_2, h_2, 100):\ 30$
$(d_3, h_2, 100):\ 20$
$(d_4, h_2, 100):\ 10$
$(d_5, h_2, 100):\ 40$

Example

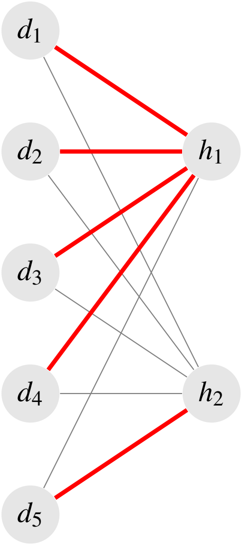
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$

$(d_1, h_1, 57): \ 111 \approx 57 \cdot 1.95$
$(d_2, h_1, 50): \ 98 = 50 \cdot 1.96$
$(d_3, h_1, 42): \ 83 \approx 42 \cdot 1.98$
$(d_4, h_1, 55): \ 110 = 55 \cdot 2$
$(d_5, h_1, 50): \ 101 = 50 \cdot 2.02$

$B_{h_2} = 100$

$(d_1, h_2, 100): \ 50$
$(d_2, h_2, 100): \ 30$
$(d_3, h_2, 100): \ 20$
$(d_4, h_2, 100): \ 10$
$(d_5, h_2, 100): \ 40$

Example

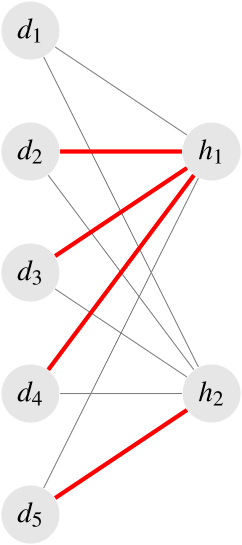
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$

$(d_1, h_1, 57): \; 111 \approx 57 \cdot 1.95$
$(d_2, h_1, 50): \; 98 = 50 \cdot 1.96$
$(d_3, h_1, 42): \; 83 \approx 42 \cdot 1.98$
$(d_4, h_1, 55): \; 110 = 55 \cdot 2$
$(d_5, h_1, 50): \; 101 = 50 \cdot 2.02$

$B_{h_2} = 100$

$(d_1, h_2, 100): \; 50$
$(d_2, h_2, 100): \; 30$
$(d_3, h_2, 100): \; 20$
$(d_4, h_2, 100): \; 10$
$(d_5, h_2, 100): \; 40$

Example

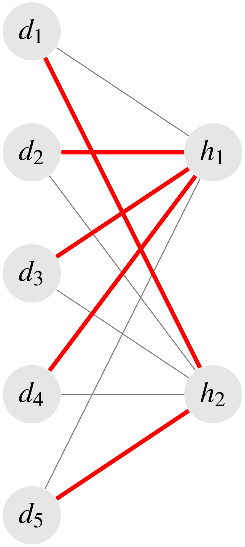
$(d_1, h_1, 57) \succ_{d_1}$ $(d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$

$(d_1, h_1, 57): \; 111 \approx 57 \cdot 1.95$
$(d_2, h_1, 50): \; 98 = 50 \cdot 1.96$
$(d_3, h_1, 42): \; 83 \approx 42 \cdot 1.98$
$(d_4, h_1, 55): \; 110 = 55 \cdot 2$
$(d_5, h_1, 50): \; 101 = 50 \cdot 2.02$

$B_{h_2} = 100$

$(d_1, h_2, 100): \; 50$
$(d_2, h_2, 100): \; 30$
$(d_3, h_2, 100): \; 20$
$(d_4, h_2, 100): \; 10$
$(d_5, h_2, 100): \; 40$

Example

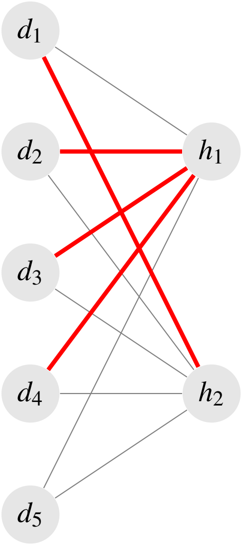
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$

$(d_1, h_1, 57): \ 111 \approx 57 \cdot 1.95$
$(d_2, h_1, 50): \ 98 = 50 \cdot 1.96$
$(d_3, h_1, 42): \ 83 \approx 42 \cdot 1.98$
$(d_4, h_1, 55): \ 110 = 55 \cdot 2$
$(d_5, h_1, 50): \ 101 = 50 \cdot 2.02$

$B_{h_2} = 100$

$(d_1, h_2, 100): \ 50$
$(d_2, h_2, 100): \ 30$
$(d_3, h_2, 100): \ 20$
$(d_4, h_2, 100): \ 10$
$(d_5, h_2, 100): \ 40$

Example

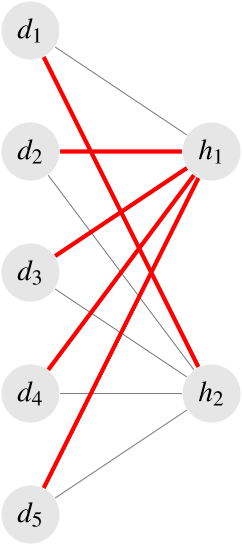
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$

$(d_1, h_1, 57): \; 111 \approx 57 \cdot 1.95$

$(d_2, h_1, 50): \; 98 = 50 \cdot 1.96$

$(d_3, h_1, 42): \; 83 \approx 42 \cdot 1.98$

$(d_4, h_1, 55): \; 110 = 55 \cdot 2$

$(d_5, h_1, 50): \; 101 = 50 \cdot 2.02$

$B_{h_2} = 100$

$(d_1, h_2, 100): \; 50$

$(d_2, h_2, 100): \; 30$

$(d_3, h_2, 100): \; 20$

$(d_4, h_2, 100): \; 10$

$(d_5, h_2, 100): \; 40$

Example

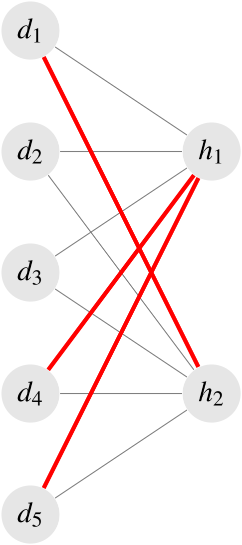
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$

$(d_1, h_1, 57): \; 111 \approx 57 \cdot 1.95$
$(d_2, h_1, 50): \; 98 = 50 \cdot 1.96$
$(d_3, h_1, 42): \; 83 \approx 42 \cdot 1.98$
$(d_4, h_1, 55): \; 110 = 55 \cdot 2$
$(d_5, h_1, 50): \; 101 = 50 \cdot 2.02$

$B_{h_2} = 100$

$(d_1, h_2, 100): \; 50$
$(d_2, h_2, 100): \; 30$
$(d_3, h_2, 100): \; 20$
$(d_4, h_2, 100): \; 10$
$(d_5, h_2, 100): \; 40$

Example

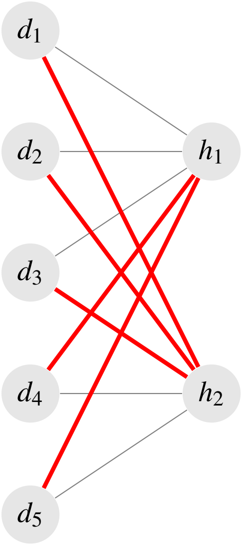
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$

$(d_1, h_1, 57): \; 111 \approx 57 \cdot 1.95$
$(d_2, h_1, 50): \; 98 = 50 \cdot 1.96$
$(d_3, h_1, 42): \; 83 \approx 42 \cdot 1.98$
$(d_4, h_1, 55): \; 110 = 55 \cdot 2$
$(d_5, h_1, 50): \; 101 = 50 \cdot 2.02$

$B_{h_2} = 100$

$(d_1, h_2, 100): \; 50$
$(d_2, h_2, 100): \; 30$
$(d_3, h_2, 100): \; 20$
$(d_4, h_2, 100): \; 10$
$(d_5, h_2, 100): \; 40$

Example

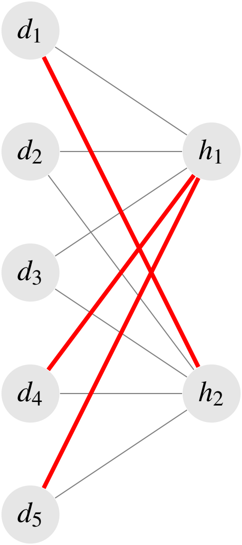
$(d_1, h_1, 57) \succ_{d_1} (d_1, h_2, 100)$

$(d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100)$

$(d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100)$

$(d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100)$

$(d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50)$



$B_{h_1} = 100$

$(d_1, h_1, 57): \; 111 \approx 57 \cdot 1.95$
$(d_2, h_1, 50): \; 98 = 50 \cdot 1.96$
$(d_3, h_1, 42): \; 83 \approx 42 \cdot 1.98$
$(d_4, h_1, 55): \; 110 = 55 \cdot 2$
$(d_5, h_1, 50): \; 101 = 50 \cdot 2.02$

$B_{h_2} = 100$

$(d_1, h_2, 100): \; 50$
$(d_2, h_2, 100): \; 30$
$(d_3, h_2, 100): \; 20$
$(d_4, h_2, 100): \; 10$
$(d_5, h_2, 100): \; 40$

Outline

1. Model
2. Generalized Deferred Acceptance Mechanism
3. Mechanisms
   - Strategy-proof mechanism
   - Not strategy-proof mechanism
4. Conclusion

Main results

Theorem

$0 < \forall\alpha < \forall\beta < 1$, $\exists$ *market* $(D, H, X, \succ_D, f_H, B_H)$ *such that*

- $x_W \leq \beta \cdot B_{x_H}$ $(\forall x \in X)$ *and*
- $\nexists$ B'_H*-stable matching if* $B_h \leq B'_h \leq (1+\alpha)B_h$ $(\forall h \in H)$

Theorem

$\exists$*mechanism such that*

- *strategy-proof for doctors and*
- *it provides a* B'_H*-stable matching such that*

$$B_h \leq B'_h \leq \left(\max_{x \in X_h} x_W\right) \cdot \left\lceil \frac{B_h}{\min_{x \in X_h} x_W} \right\rceil \quad (\forall h \in H)$$

Theorem

$\exists$*mechanism that provides a* B'_H*-stable matching such that*

$$B_h \leq B'_h < B_h + \max_{x \in X_h} x_W \quad (\forall h \in H)$$

Other results for special cases

Theorem

If $f_h(X'_h) = w_h(X'_h)$, *∃mechanism such that*

- *strategy-proof for doctors and*
- *it provides a* B'_H*-stable matching such that*
$$B_h \leq B'_h < B_h + \max_{x \in X_h} x_W \quad (\forall h \in H)$$

Theorem

If $f_h(X'_h) = w_h(X'_h)$, *∃mechanism that gives a* B'_H*-stable matching such that*
$$B_h \leq B'_h < 1.5 \cdot B_h \quad (\forall h \in H)$$

Theorem

If $f_h(X'_h) = |X'_h|$, *∃mechanism such that*

- *strategy-proof for doctors and*
- *it provides a stable matching*