Computing Least Cores of Supermodular Cooperative Games (AAAI’17)

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An example of the cooperative game

- Given (V, ν)
  - $V = \{\quad\}$
  - $\nu(\quad) = 15$
  - $\nu(\quad) = 25$
  - $\nu(\quad) = 30$
  - $\nu(\quad) = 30$
  - $\nu(\quad) = 50$
  - $\nu(\quad) = 50$ ← violate
  - $\nu(\quad) = 80$

$\nu(\quad) = 80$

Find a value division

- $x(\quad) = 15$
- $x(\quad) = 35$
- $x(\quad) = 30$

$x(\quad) + x(\quad) \leq \nu(\quad)$

leave from the cooperation

An example of the cooperative game

- Given (V, ν)
  - $V = \{\ \ \}$
  - $\nu(\ ) = 15$
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  - $\nu(\ \ \ ) = 80$

$\nu(\ \ \ ) = 80$

Find a value division

- $x(\ ) = 15$
- $x(\ ) = 30$
- $x(\ ) = 35$

No violation!

No one leaves

A goal of the cooperative game

Finding a core can be described as the following program

$$\begin{array}{lll} \text{find} & \boldsymbol{x} & \\ \text{s.t.} & \boldsymbol{x}(S) \geq \nu(S) & \forall \emptyset \subsetneq S \subsetneq V \\ & \boldsymbol{x}(V) = \nu(V) & \\ & \boldsymbol{x}(v) \geq 0 & \forall v \in V. \end{array}$$

- A value division $\boldsymbol{x} \in \mathbb{R}^V_+$ if $\boldsymbol{x}(V) = \nu(V)$
- A value division is in the core if $\boldsymbol{x}(S) \geq \nu(S)$ for all $S \subseteq V$

Finding a core of a cooperative game is **NP-complete**

⇨ **Supermodular cooperative game**

Supermodular cooperative game

- Supermodular cooperative game is a cooperative game where a characteristic function v is *supermodular*

  A function $f : 2^V \to \mathbb{R}$ is called supermodular if
  $f(S) + f(T) \leq f(S \cap T) + f(S \cup T)$
  for all $S, T \subseteq V$, and $S \subseteq T$

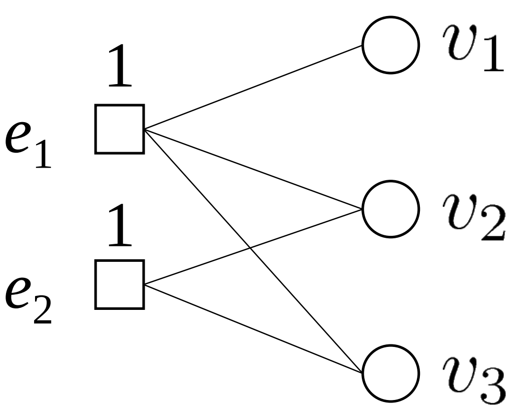
- Applications
  - **Induced subgraph game** [Deng and Papadimitriou 1994]
  - **Airport game** [Littlechild and Owen 1973]
  - **Bidder collusion game** [Graham, Marshall and Richard 1990]
  - **Multicast tree game** [Feigenbaum, Papadimitriou and Shenker 2001]
  - **Bankruptcy game** [O'Neill 1982]

An application of the supermodular cooperative game

Induced subgraph game

- Weighted hypergraph: $G = (V, E, w)$
- Characteristic function $\nu : 2^V \to \mathbb{R}_+$ is the total weight of hyperedges $e \in E$ such that every vertex in S belongs to e

Example:

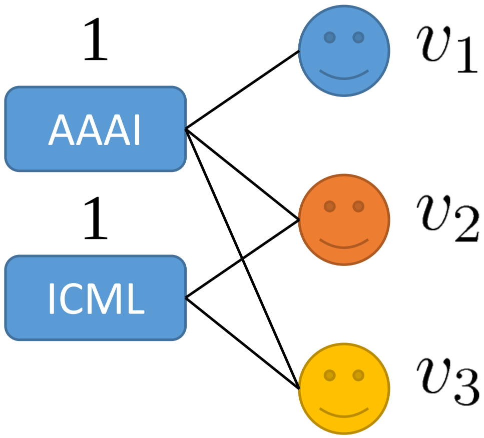


$$\nu(v_1) = \nu(v_2) = \nu(v_3) = 0,$$
$$\nu(v_1, v_2) = 0, \ \nu(v_2, v_3) = 1, \ \nu(v_1, v_3) = 0,$$
$$\nu(v_1, v_2, v_3) = 2.$$

An example of **induced subgraph game**

In a certain K-project…

- Professor K gives bonuses depending on the number of accepted top conference-papers
- Want to find a bonus division in the core



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※この例はフィクションであり、実在する人物、団体等とは一切関係ありません

Motivation

- Property of supermodular cooperative game
  - A value division called <u>*Shapley value* is **always** in the core</u>

- Focus on more general solution concepts than the core
  - Want to find the *stablest* value division
    - Strong least core $\boldsymbol{x}(S) - \nu(S) \geq \epsilon \quad \forall \emptyset \subsetneq S \subsetneq V$
    - Weak least core $\boldsymbol{x}(S) - \nu(S) \geq \epsilon |S| \quad \forall \emptyset \subsetneq S \subsetneq V$

Goal of our work

- Goal is to find **the strong and weak least core values** of <u>supermodular cooperative game</u>
- The strong and weak least core values is obtained by solving the following LP

$$
\begin{array}{lll}
\min & -\epsilon & \\
\text{s.t.} & \boldsymbol{x}(S) \geq \nu(S) + \epsilon & \forall \emptyset \subsetneq S \subsetneq V \\
& \boldsymbol{x}(V) = \nu(V) & \\
& \boldsymbol{x}(v) \geq 0 & \forall v \in V,
\end{array}
$$

where ν is supermodular

No computational analysis about the strong and weak least core values of supermodular cooperative game exists

Idea of our approach (1)

- Instead of solving the optimization problem, fix $\epsilon > 0$ and consider its feasibility
- Define a function $f_\epsilon : 2^V \to \mathbb{R}$ as

$$f_\epsilon(S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ -\nu(S) - \epsilon & \text{if } \emptyset \subsetneq S \subsetneq V, \\ -\nu(V) & \text{if } S = V. \end{cases}$$

(V, ν) is a supermodular game

- Solve the following feasibility problem

$$\begin{array}{lll} \text{find} & \mathbf{x} & \\ \text{s.t.} & \mathbf{x}(S) \leq f_\epsilon(S) & \forall \emptyset \subsetneq S \subsetneq V \\ & \mathbf{x}(V) = f_\epsilon(V) & \\ & \mathbf{x}(v) \leq 0 & \forall v \in V. \end{array}$$

Idea of our approach (2)

- To check the feasibility…

$$
\begin{array}{lll}
\text{find} & \mathbf{x} & \\
\text{s.t.} & \mathbf{x}(S) \le f_\epsilon(S) & \forall \emptyset \subsetneq S \subsetneq V \\
& \mathbf{x}(V) = f_\epsilon(V) & \\
& \mathbf{x}(v) \le 0 & \forall v \in V.
\end{array}
$$

- Define $p_\epsilon = \max\{\boldsymbol{x}(V) \mid \boldsymbol{x} \in P(f_\epsilon)\}$, where $P(f) = \{\boldsymbol{x} \in \mathbb{R}^V \mid \boldsymbol{x}(S) \le f(S) \text{ for all } \emptyset \subsetneq S \subsetneq V\}$

- The above program is feasible iff $p_\epsilon \ge f_\epsilon(V)$
    - p_ϵ can be computed in polynomial time (Frank and Tardos 1988; Naitoh and Fujishige 1992)

Idea of our approach (3)

$S, T \subseteq V$ are **crossing** if $S \cap T \neq \emptyset, S \setminus T \neq \emptyset, T \setminus S \neq \emptyset$, and $S \cup T \neq V$. A function $f : 2^V \to \mathbb{R}$ is called **crossing submodular** if

$$f(S) + f(T) \geq f(S \cap T) + f(S \cup T)$$

holds for any crossing $S, T \subseteq V$.

$$f_\epsilon(S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ -\nu(S) - \epsilon & \text{if } \emptyset \subsetneq S \subsetneq V, \\ -\nu(V) & \text{if } S = V. \end{cases}$$

is crossing submodular

Idea of our approach (4)

To compute p_ϵ…

Theorem 1 (Fugishige 1984) *Let $f : 2^V \to \mathbb{R}$ be a crossing submodular function and let $p = \max\{\boldsymbol{x}(V) \mid \boldsymbol{x} \in P(f)\}$. Let $q, r \in \mathbb{R}$ be defined as*

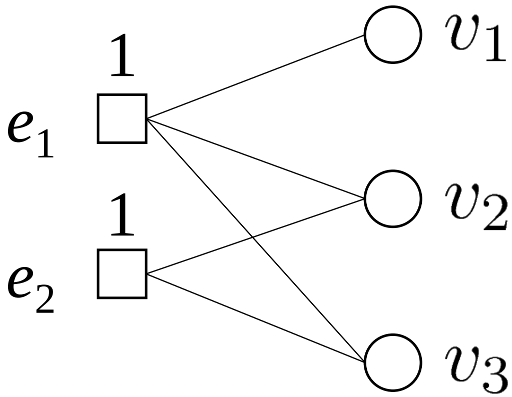
$$q = \min_{\mathcal{S} \in \mathcal{P}_2(V)} \sum_{S \in \mathcal{S}} f(S),$$

$$r = \min_{\mathcal{S} \in \bar{\mathcal{P}}_3(V)} \frac{1}{|\mathcal{S}| - 1} \sum_{S \in \mathcal{S}} f(S).$$

Then, we have $p = \min\{q, r\}$.

By the theorem, characterize the strong and weak least core values of the supermodular cooperative game

Induced subgraph game



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- The <u>strong least core value</u>

$$\epsilon^* = -\min_{\mathcal{S} \in \mathcal{P}_2} \frac{c(\mathcal{S})}{|\mathcal{S}|}$$

How well the hypergraph is clustered by the partition $\mathcal{S}$

$\mathcal{P}_2$: the familiy of all partitions $\mathcal{S}$ of V at least 2 sets

$c(\mathcal{S})$: cut weight of a partition $\mathcal{S}$ of V

- A <u>value division</u> in the strong least core
  - Use the **ellipsoid method** to find the value division

Conclusion

- Goal:
  To find the **strong least core** and the **weak least core** of the **supermodular cooperative game**

- Contributions:
  - Provide theoretical characterizations of the strong and the weak least core values
  - Derive explicit concise formulations of the **induced subgraph game** and the **airport game**