ERATO 感謝祭 Season II

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（JST, ERATO, 河原林巨大グラフプロジェクト）

Joint work with

河原林 健一 (NII & JST, ERATO)

Today's topics

- Spanning closed walks and TSP in 3-connected planar graphs, JCTB 109 (2014) 1-33.

- 4-connected projective-planar graphs are hamiltonian-connected, JCTB 112 (2015) 36-69.

Journal of Combinatorial Theory Series B (JCTB)

：組合せ論の雑誌，Series B は主にグラフ理論

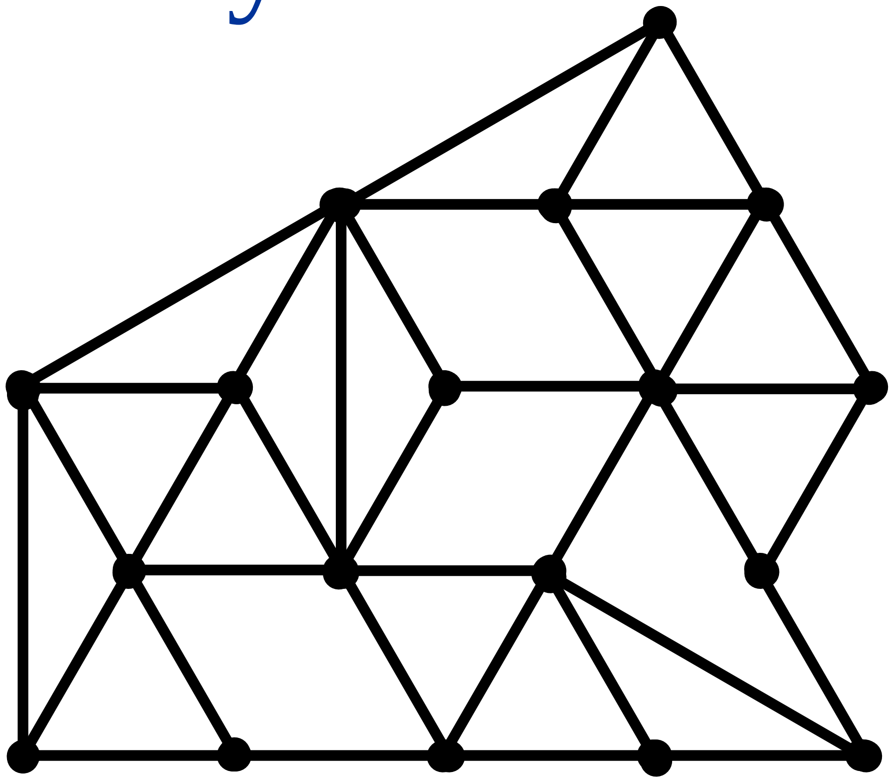
Hamiltonian cycles

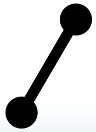
Hamiltonian cycles in G

$\Updownarrow$

Cycles passing

$\forall$ vertices in G



$\bullet \in V(G)$  $\in E(G)$

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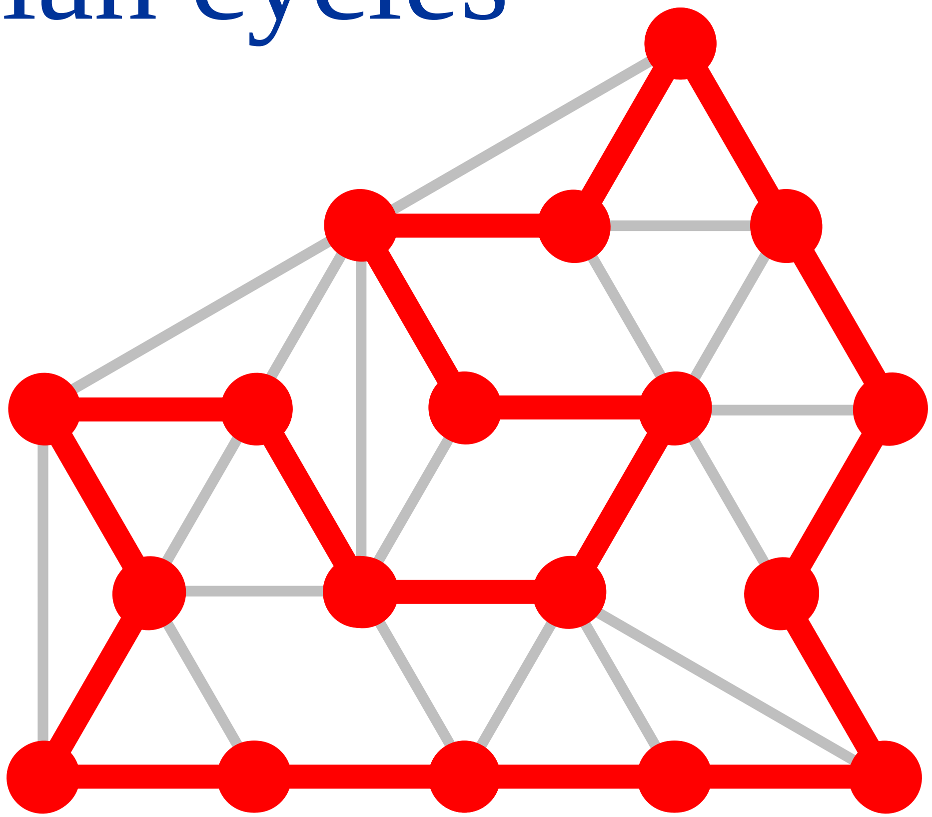

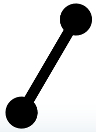

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Hamiltonicity of plane graphs

Tait (1884) :

$\forall$ cubic maps have Hamiltonian cycle

$\Downarrow$

$\forall$ plane graphs have a 4-coloring **True (4-color thm.)**

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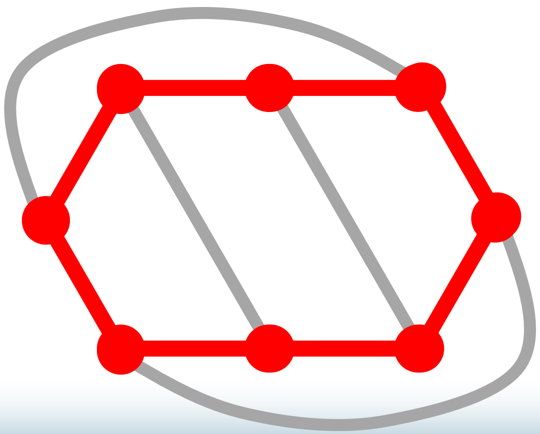

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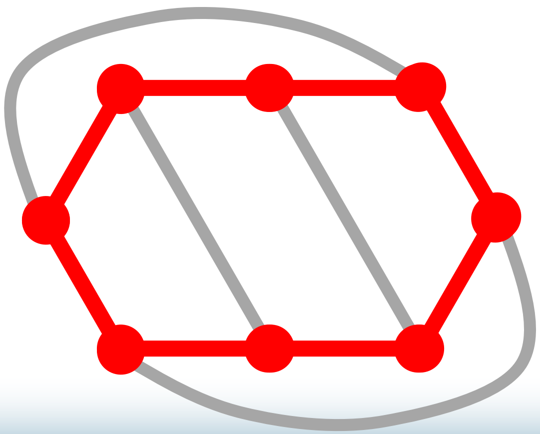

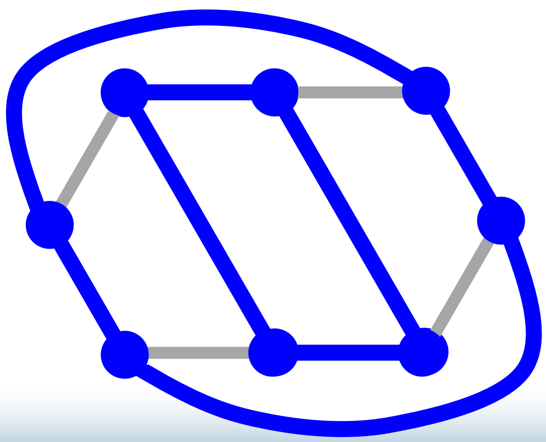

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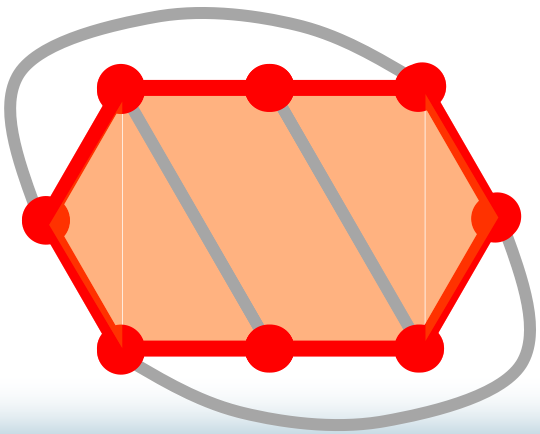

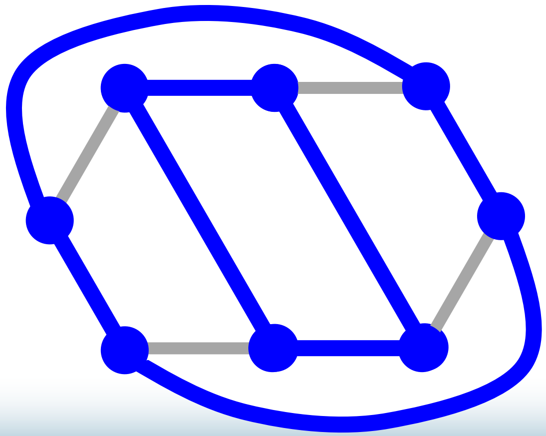

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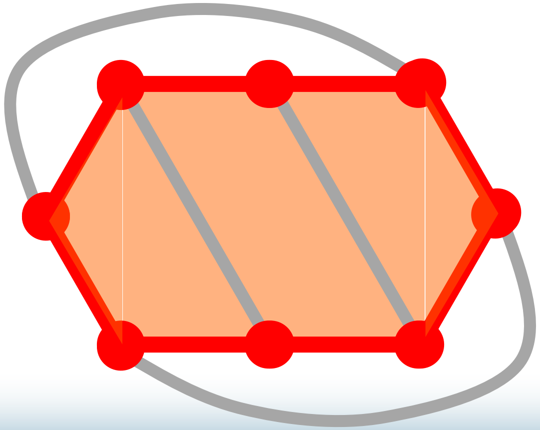

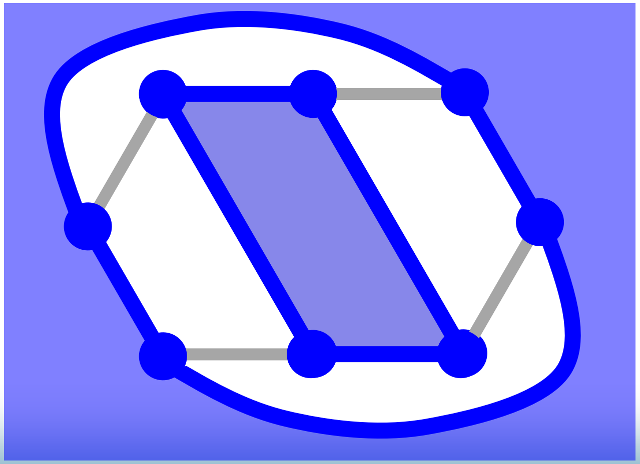

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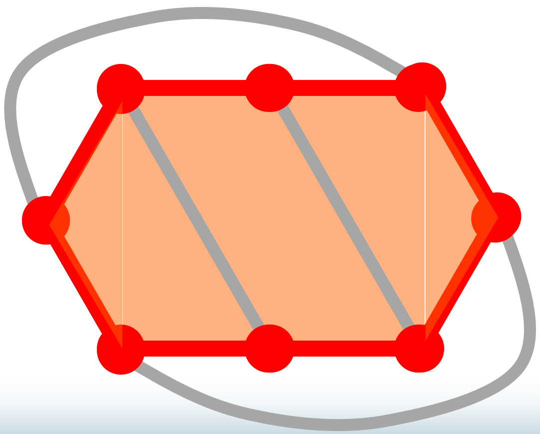

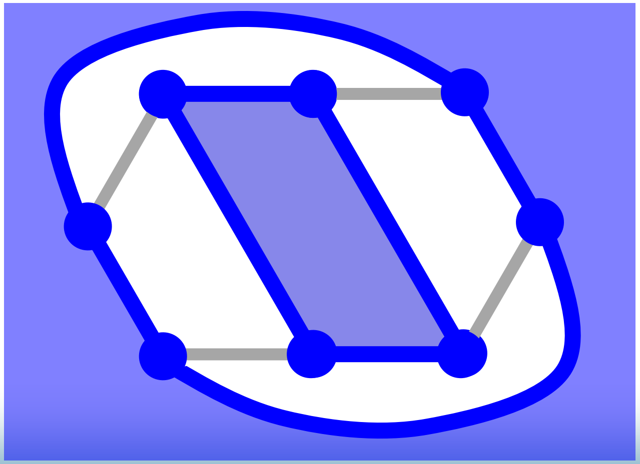

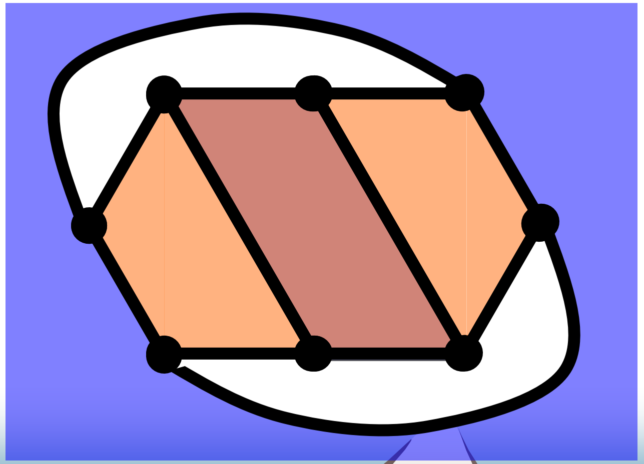

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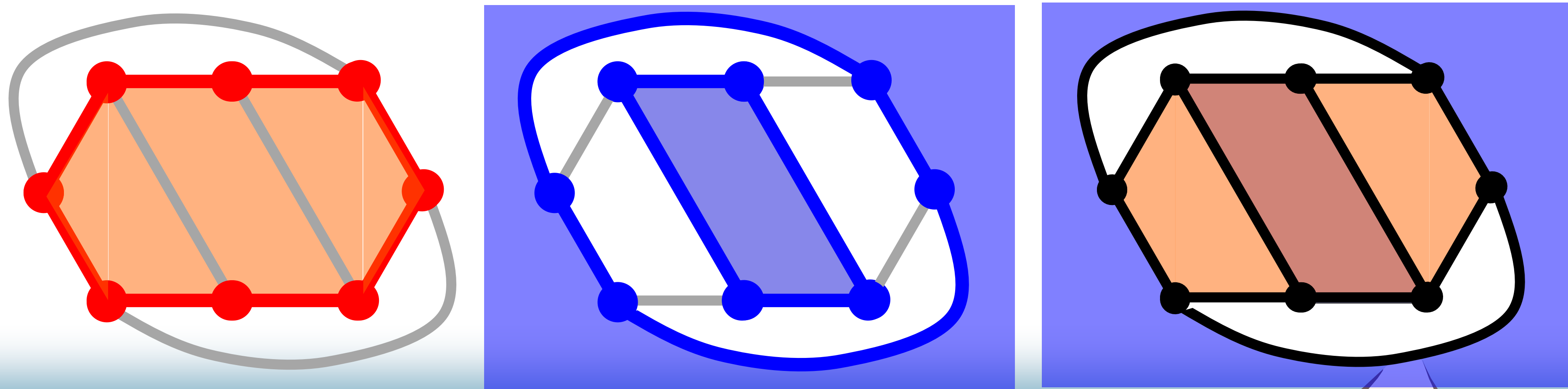

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Other application

- TSP (Travelling Salesman Problem)
- VLSI (Very Large Scale Integration) layout

Hamiltonian cycles

<u>Theorem (Tutte, `56)</u>

$\forall$ 4-connected plane graph has a Hamiltonian cycle.

G : k-conn. $\Leftrightarrow \forall S \subset V(G)$ with $|S| \leq k-1$
$G - S$ is connected

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There still remain several interesting conjectures in this area.

- ✓ The 3-connected case (Goemans Conj.)
- ✓ The case of graphs on projective plane (Dean Conj.)

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Find ``good" structures which are close to Hamiltonian cycles in 3-conn. plane graphs

Including plane triangulations

Hamiltonian cycles

Remark:

∃ 2-conn. plane graphs

which are far from being Hamiltonian

Ex. Complete bipartite graph $K_{2,m}$

Problem

Find ``good" structures which are close to Hamiltonian cycles in 3-conn. plane graphs

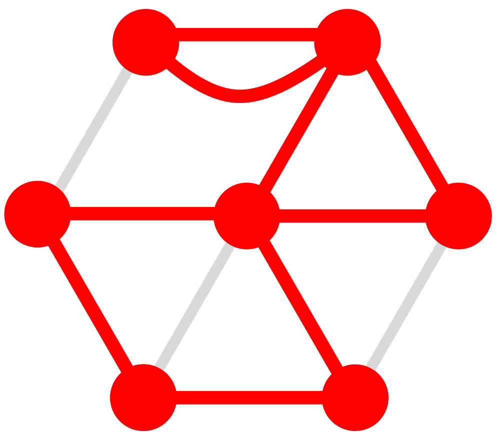
Including plane triangulations

Spanning closed walks

Here we focus on a spanning closed walk with short length.

SCW : Cycle which can pass thr. vertices many times

A Hamiltonian cycle = a SCW of length $|G|$



SCW of length 9

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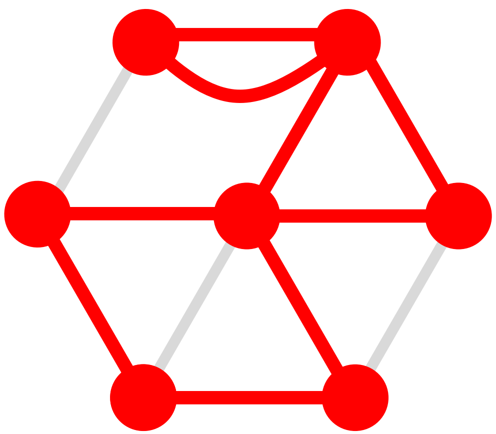

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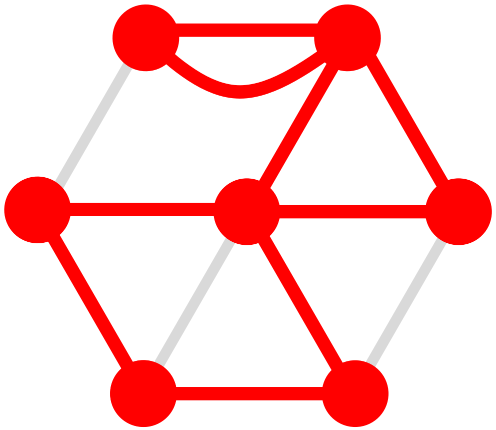

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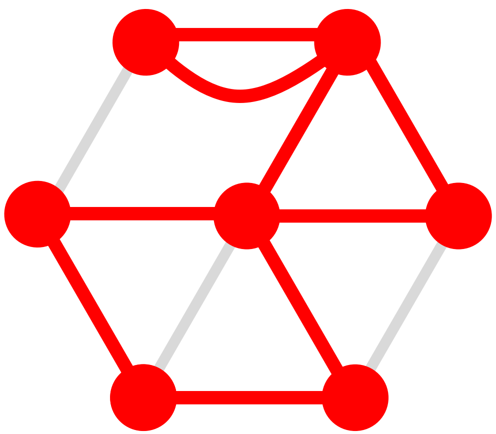

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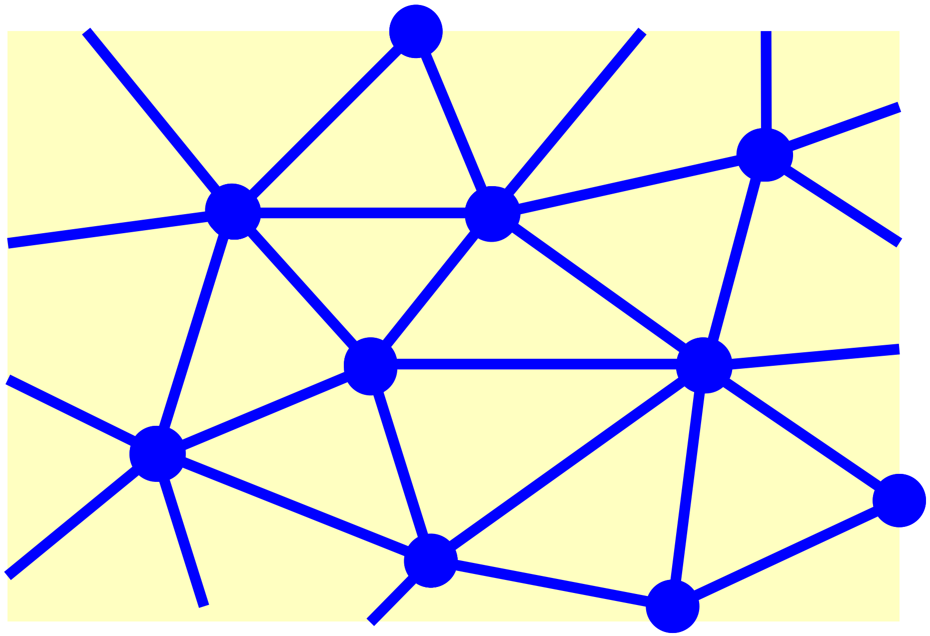
Main Theorem

$O(n^2)$-time algorithm

$\forall$3-conn. plane graph has a SCW of length $\leq \frac{4|G|-4}{3}$.

Best possibility



H : Plane triangulation

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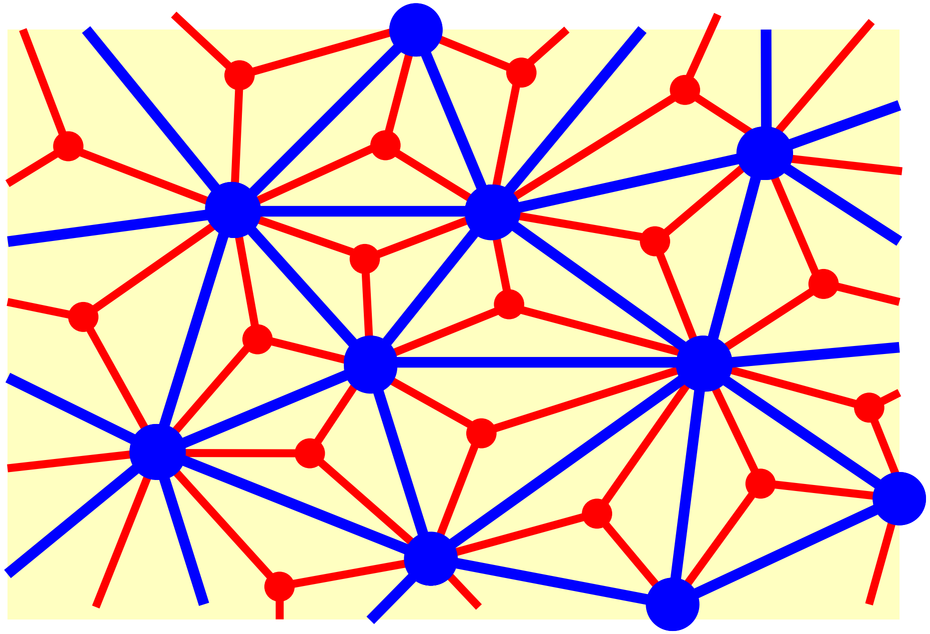

H : Plane triangulation

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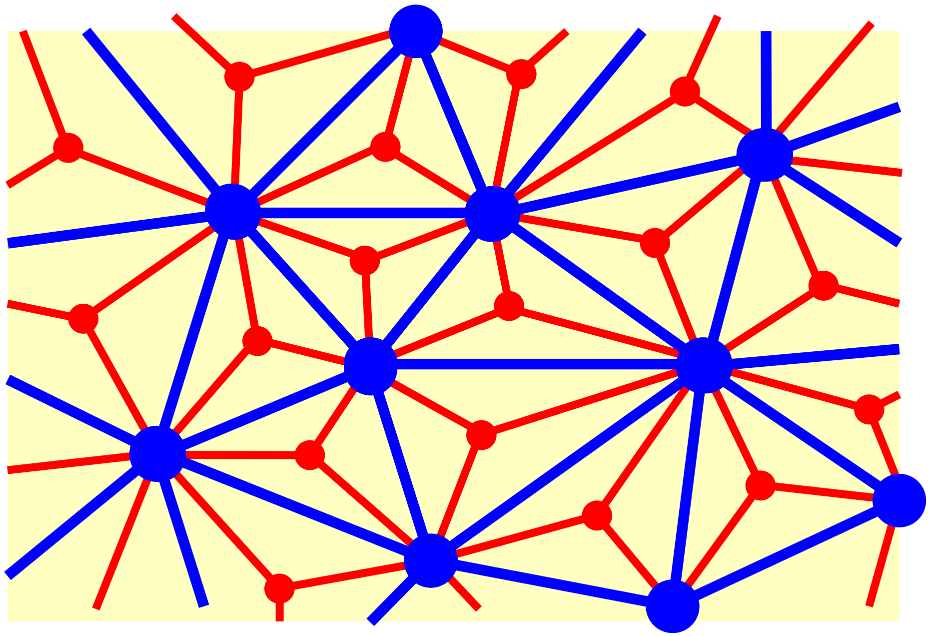

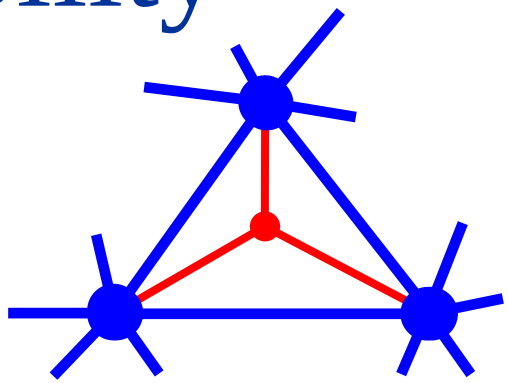

To span each added vertex,
we need the length ≥ 2

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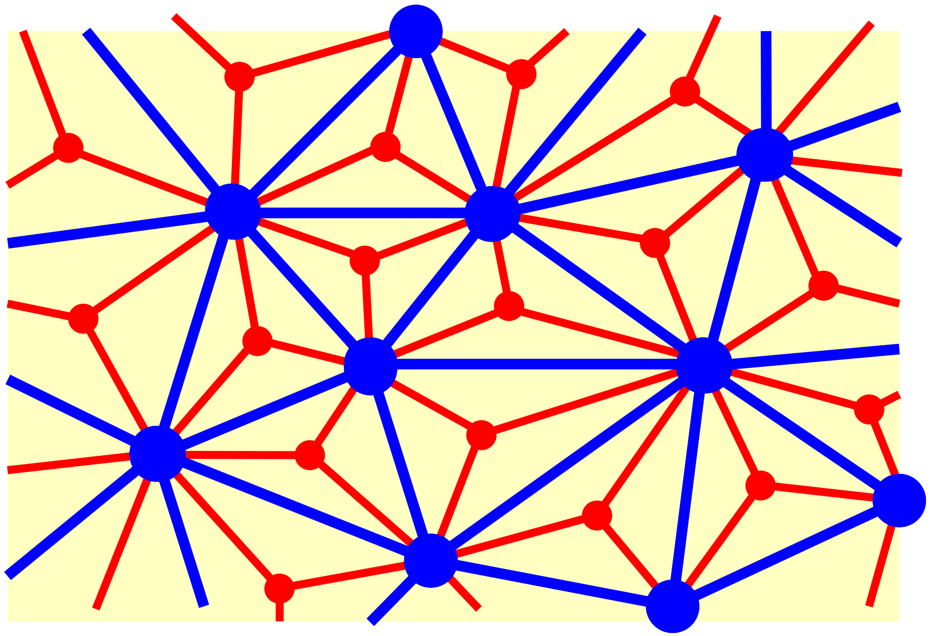

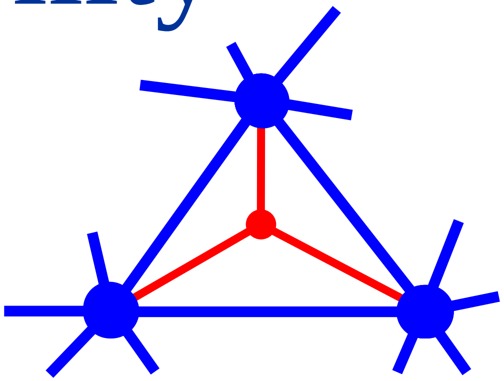

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For every SCW,
the length of it is

$$\geq 2|V(G) - V(H)|$$
$$= 2(2|V(H)| - 4) \quad = \frac{4|V(G)| - 8}{3}$$

GTSP for plane graphs

Graph-Traveling Salesman Problem (GTSP) :

Input: a (undirected) graph G of order n

Want: a shortest Hamilton cycle of the complete graph K_n,

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GTSP $\min \sum_{e \in E(G)} x(e)$

sub. to $x(\delta(S)) \geq 2$ for $\forall S \subset V(G)$

$x(\delta(\{v\})) = 2$ for $\forall v \in V(G)$

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SCW condition

Integer restriction

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4/3-Conjecture (Goemans '95)

$\forall G$: (connected) graph

$$\frac{\text{OPT(GTSP)}}{\text{OPT(SER)}} \leq \frac{4}{3}$$

SER : Subtour Elimination Relaxation

(Linear prog. relax. of GTSP)

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`` $\leq$ 3/2 '' is true for $\forall$ graph (Wolsey '80, Shmoys, Williamson '90)

Remark: OPT(GTSP) = length of a shortest SCW in G,
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Conjectured by Goemans `07

Cor. of Main Thm.

4/3-Conjecture is true for 3-conn. plane graphs.

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For $\forall$ pair of vertices,
$\exists$H-path between them

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$\Rightarrow$ $\exists$Hamilton cycle

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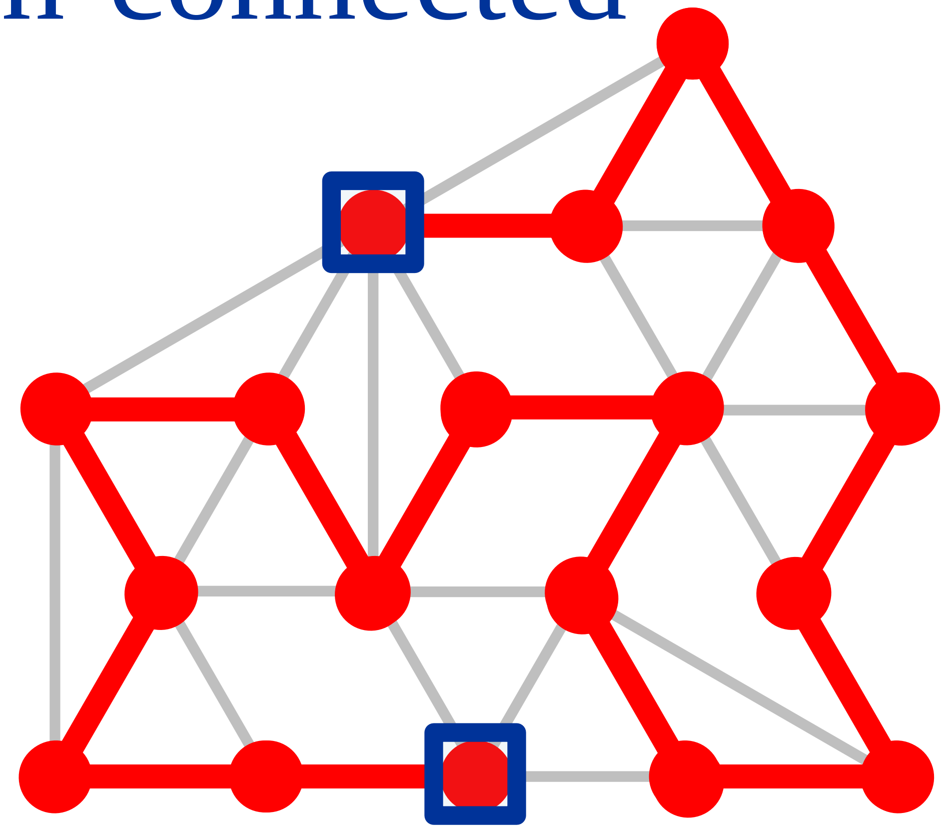

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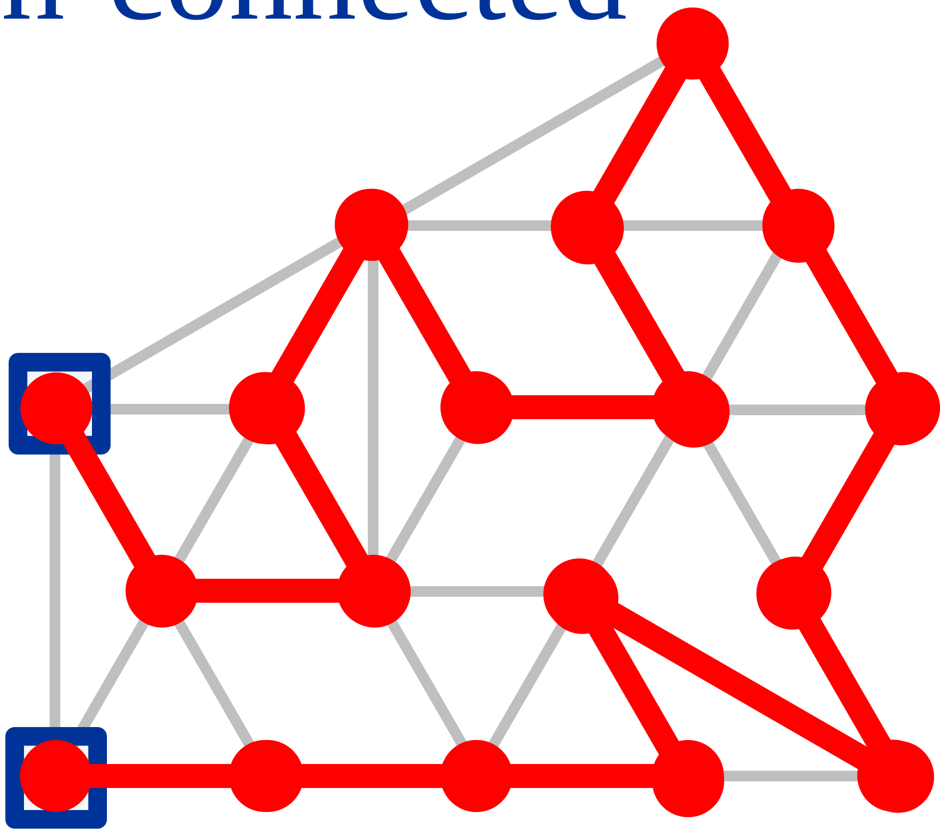

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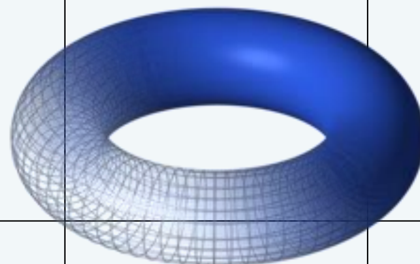
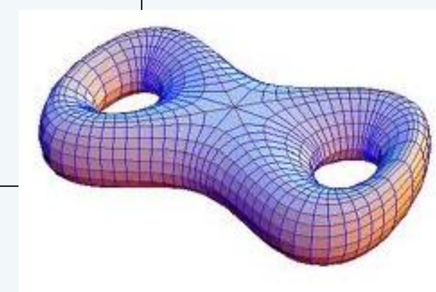
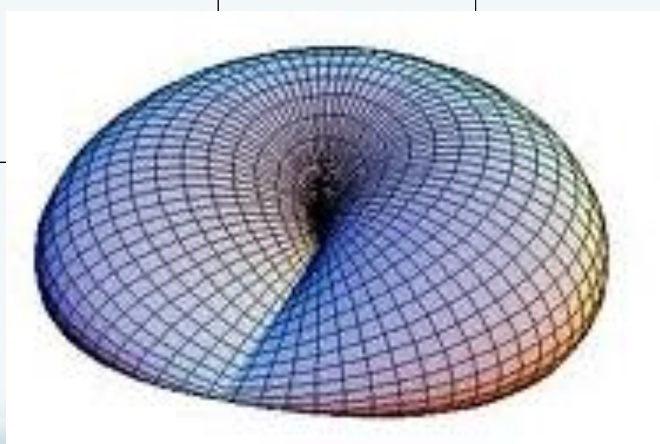
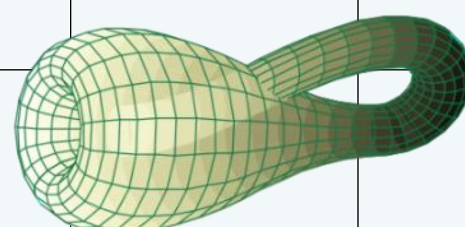
Surfaces

	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$	K-bottle $\chi = 0$	N_3 $\chi = -1$	Others $\chi < -1$









4-connected graphs on surfaces

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H-path	○	○	○ Thomas, Yu & Zang ('05)			×
H-cycle	○ Tutte ('56)	○ Thomas & Yu ('94)			×	×
H-conn	○ Thomassen ('83)		×	×	×	×

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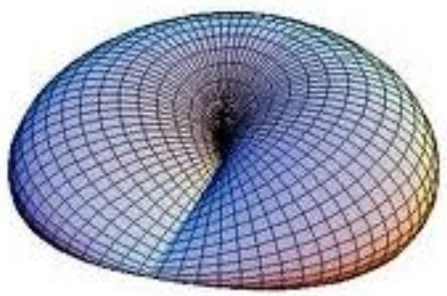
4-conn. graphs on projective plane

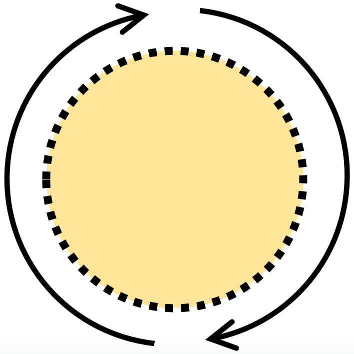
Main Thm.

$\forall G$: 4-connected graphs on the projective plane

$\Rightarrow$ Hamiltonian-connected

Projective plane





The view of Algorithm

Our proof implies the following $O(n^2)$ -algorithm

Input : A 4-conn. Projective-planar graph G of order n

2 vertices x, y of G

Output : Hamilton path (Tutte path) in G between x, y

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Chiba & Nishizeki (`89) :

$\exists$ $O(n)$ -time algorithm

to find Hamilton cycle in 4-conn. plane graph of order n

Summary

Theorem (Tutte, `56)

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There still remain several interesting conjectures in this area.

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Thank you for your attention